B-Mixing Basics
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1 Introduction

For the CKM experiment, we want to know how well our extraction of V_{td} would compare to that of B-mixing. The measurements of Δm_s and Δm_d in $B_s^0 - \bar{B}^0{}_s$ and $B_d^0 - \bar{B}^0{}_d$ mixing respectively would give the following:

$$\frac{\Delta m_d}{\Delta m_s} = \frac{f_{B_d}^2 B_{B_d} m_{B_d} |V_{td}|^2}{f_{B_s}^2 B_{B_s} m_{B_s} |V_{ts}|^2} \tag{1}$$

Hence, one can extract V_{td} in a manner that is claimed to be largely free of theoretical uncertainties. Only B_d mixing, however, has been observed.

I am interested here in how the factors f_B and B_B come about. These have to be calculated with non-perturbative techniques. They are important, since they limit the uncertainty of $|V_{td}|/|V_{ts}|$ to 7% of itself in the most aggressive theoretical estimate. Various techniques such as vaccuum insertion, MIT Bag, and lattice gauge, have been used to evaluate f_B and B_B. I will not describe these techniques here.

The purpose of this note is to review, at a basic level, how f_B and B_B play themselves into Δm. Some simplifying assumptions are made, but then I give the full expression in section 5. The calculation is shown for $B_d^0 - \bar{B}_d^0$ mixing. The last section summarizes the latest estimates of f_B and B_B for both $B_d^0 - \bar{B}_d^0$ and $B_s^0 - \bar{B}_s^0$ mixing.

2 Δm Calculation

Three distinct steps are needed to calculate Δm. First, one needs time-dependent perturbation theory to derive an expression for Δm. The expression can be associated with various box diagrams. Next, the box diagrams are reduced to an effective current-current operator. Finally, one has to evaluate the expectation value of these current-current operators. It is this last step in which f_B and B_B appears.

We have from time-dependent perturbation theory (see appendix):

$$\Delta m = B_L - B_S = 2\Sigma_n \frac{< \bar{B}^0 | H_{WK} | n >< n | H_{WK} | B^0 >}{m_B - E_n} \tag{2}$$

$|n>$ represents all possible intermediate states, and the term $\frac{1}{m_B - E_n}$ represents the propagator of the state $|n>$.

At this point, one can identify each term in the sum over $|n>$ as being *equal* to the amplitude of various graphs. The lowest order graph in the sum is shown in figure 1. For completeness, I give below the amplitude for the (free) u-quark intermediate

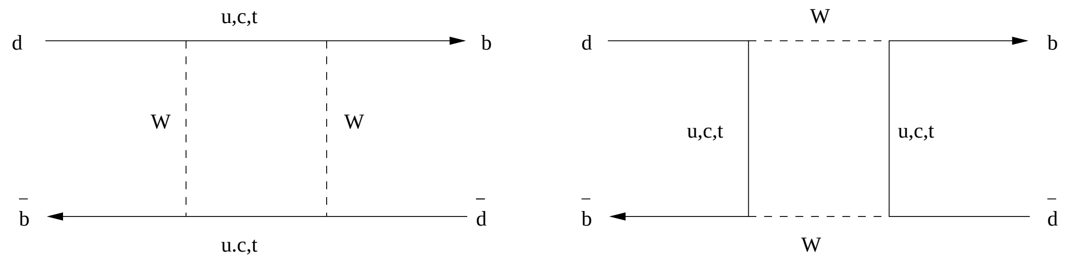

Figure 1: Lowest order diagrams corresponding to equation 2.

state (see Commins and Bucksbaum):

$$\begin{aligned}
\Delta m &= 2 < \bar{B}^0 | M_u | B^0 > & (3)\\
M_u &= 2i(V_{ub}^* V_{ud})^2 (\frac{-i 2^{5/4} m_W \sqrt{G_F}}{2\sqrt{2}})^2 \quad \cdot & (4)\\
& \int \frac{d^4 k}{(2\pi)^4} \bar{u}_b \gamma_\lambda (1-\gamma_5) \frac{\not{k} + m_u}{k^2 - m_u^2} \quad \cdot & (5)\\
& \gamma_\rho (1-\gamma_5) u_d \bar{\nu}_b \gamma_\alpha (1-\gamma_5) \frac{\not{k} + m_u}{k^2 - m_u^2} \gamma_\sigma (1-\gamma_5) \nu_d \quad \cdot & (6)\\
& (-i \frac{g^{\lambda\sigma} - k^\lambda k^\sigma / m_W^2}{k^2 - m_W^2})(-i \frac{g^{\alpha\rho} - k^\alpha k^\rho / m_W^2}{k^2 - m_W^2}) & (7)
\end{aligned}$$

Here, the integral is performed over the u-quark 4-momenta, and one recognizes the two u-quark and W propagators. u_q, ν_q are Dirac spinors for the quark q. After integrating over the u-quark and W degrees of freedom, M_u reduces to an *effective* current-current operator. Taking the expectation value, one gets:

$$< \bar{B}^0 | M_u | B^0 > = \frac{G_F^2}{4\pi^2} m_W^2 (V_{ub}^* V_{ud})^2 < \bar{B}^0 | \bar{b} \gamma_\mu (1-\gamma_5) d \cdot \bar{b} \gamma^\mu (1-\gamma_5) d | B^0 > \qquad (8)$$

Between the operators d and $\bar{b}$, we may insert a factor $1 = \Sigma_n |n>< n| \approx |0>< 0|$. This step is called the "vaccuum insertion approximation". One has:

$$\begin{aligned}
< \bar{B}^0 | \bar{b} \gamma_\mu (1-\gamma_5) d \cdot \bar{b} \gamma^\mu (1-\gamma_5) d | B^0 > &\approx & (9)\\
< \bar{B}^0 | \bar{b} \gamma_\mu (1-\gamma_5) d | 0 >< 0 | \bar{b} \gamma^\mu (1-\gamma_5) d | B^0 > & & (10)
\end{aligned}$$

I have ignored factors for color and intermediate states of the c and t-quark because they are not necessary for the understanding of f_B and B_B. Including them does not change the effective current-current structure of equation 8. The full expression is given in section 5.

3 f_B

The factor $< 0|\bar{b}\gamma^{\mu}(1-\gamma_5)d|B^0 >$ is related to the famous B-meson decay constant f_B. Recall that the fully leptonic B^- decay (figure 2) has the following amplitude (see Commins and Bucksbaum):

$$A = \frac{G_F}{\sqrt{2}}\bar{u}(p)\gamma_\mu(1-\gamma_5)\nu(k)[-\sqrt{2E_B} < 0|J^{\dagger\mu}|B^- >] \tag{11}$$

$$[-\sqrt{2E_B} < 0|J^{\dagger\mu}|B^- >] = if_B V_{ub} q^{\mu} \tag{12}$$

The factor $f_B V_{ub}$ can be extracted experimentally from the B-lifetime and the leptonic branching ratio. Note however, that f_B cannot be measured independent of its accompanying CKM matrix element. So an experimental extraction of f_B is only as good as the CKM matrix element uncertainty. At this point, the following identification is made:

$$\begin{aligned} < 0|J^{\dagger\mu}|B^- > &= V_{ub} < 0|\bar{b}\gamma^{\mu}(1-\gamma_5)u|B^- > &(13)\\ &= V_{ub} < 0|\bar{b}\gamma^{\mu}(1-\gamma_5)d|B^0 > &(14)\end{aligned}$$

In the first step, one associates the raising operator $J^{\dagger\mu}$ with *free* quark spinors. In the second step, one uses the assumption that the u and d quark masses are the same, i.e. the SU(3) flavor symmetry limit, and that EM interaction differences between B^0 and B^- can be ignored. So:

$$< 0|\bar{b}\gamma^{\mu}(1-\gamma_5)d|B^0 >= -\frac{1}{\sqrt{2E_B}}if_B q^{\mu} \tag{15}$$

Inserting equation 15 into equation 8, one gets ($E_B = m_B$):

$$\Delta m = \frac{G_F^2}{4\pi^2}m_W^2(V_{ub}^* V_{ud})^2 f_B^2 m_B \tag{16}$$

4 B_B

The factor B_B originates as a *correction* to the vaccuum insertion approximation. Historically, the approximation first done for $K^0 - \bar{K}^0$ mixing by Gaillard and Lee. It was pointed out that this approximation is not so good in that the one pion intermediate state contribution is significant (Shrock and Treiman). The one pion contribution is comparable in size and opposite in sign to that of the vaccuum insertion

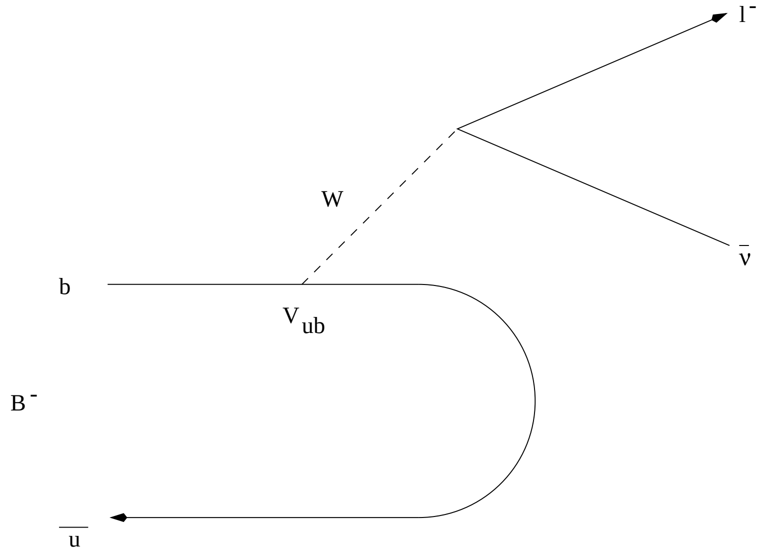

Figure 2: Fully leptonic B^- decay.

approximation. The first calculation of the correction (B_K in this case) was done using the MIT Bag Model.[1] As far as I know, no one has devised a way to measure B_B or B_K independently.

Instead of equation 10, one has to insert a correction factor B_B:

$$< \bar{B}^0|\bar{b}\gamma_\mu(1-\gamma_5)d \cdot \bar{b}\gamma^\mu(1-\gamma_5)d|B^0 > \quad = \tag{17}$$
$$B_B < \bar{B}^0|\bar{b}\gamma_\mu(1-\gamma_5)d|0 >< 0|\bar{b}\gamma^\mu(1-\gamma_5)d|B^0 > \tag{18}$$

The expression for Δm becomes:

$$\Delta m = \frac{G_F^2}{4\pi^2} m_W^2 (V_{ub}^* V_{ud})^2 f_B^2 m_B B_B \tag{19}$$

5 Full expression for Δm

When color factors and the c and t-quark intermediate states are included, the correct expression for Δm becomes:

[1] I believe this is why B_K, B_B are now called "bag factors".

$$\begin{aligned}
\Delta m &= 2Re(M_{12}) && (20)\\
M_{12} &= \frac{G_F^2}{12\pi^2} m_W^2 f_B^2 B_B m_B \cdot && (21)\\
& \quad [(V_{cb}^* V_{cd})^2 \eta_1 S(x_c) + && (22)\\
& \quad (V_{tb}^* V_{td})^2 \eta_2 S(x_t) + && (23)\\
& \quad (V_{tb}^* V_{td} V_{cb}^* V_{cd}) \eta_3 S(x_c, x_t)] && (24)\\
x_c &= m_c^2 / m_W^2 && (25)\\
x_t &= m_t^2 / m_W^2 && (26)
\end{aligned}$$

Here, $S(x_c)$, $S(x_t)$, $S(x_c, x_t)$ are the so-called Inami-Lim functions. They come about when one includes in equation 4 the c and t-quark degrees of freedom including interference. The factors η_1, η_2, and η_3 represent short-distance QCD corrections to the box diagram. To a good approximation, $S(x) = x$, so the middle term containing x_t dominates.[2] One gets (using $|V_{tb}| \approx 1$):

$$\Delta m = \frac{G_F^2}{6\pi^2} m_W^2 f_B^2 B_B m_B |V_{td}|^2 \eta_2 S(x_t) \tag{27}$$

Finally, taking the ratio $\Delta m_d / \Delta m_s$:

$$\frac{\Delta m_d}{\Delta m_s} = \frac{f_{B_d}^2 B_{B_d} m_{B_d} |V_{td}|^2}{f_{B_s}^2 B_{B_s} m_{B_s} |V_{ts}|^2} \tag{28}$$

6 f_B and B_B Estimates

The literature on f_B and B_B estimates is vast. I quote here only Kronfeld's talk at HQ98 and the latest result from the MILC collaboration. The claim is that $\frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}}$ has significantly reduced theoretical uncertainties. In principle, if $m_s = m_d$ in the flavor SU(3) symmetry limit, the ratio is unity. So again, the difference from unity has to be calculated from non-perturbative techniques. Buchalla and Buras gives the following summary in their latest preprint:

$$\frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}} = 1.14 \pm .08 \tag{29}$$

This is a 7% relative error. At this point, one could use $|V_{ts}| = A\lambda^2 = .0385 \pm .0035$ (9% relative error) and claim that $|V_{td}|$ is extracted to 11% using equation 1. However,

[2]In contrast, all three terms are important in the kaon case.

Table 1: f_{B_d} Estimates.

	MeV	stat	sys	quench
JLQCD	173	4	9	9
FNAL/UIUC	164	$^{+14}_{-11}$	8	$^{+10\%}_{-0\%}$
GLOK	147	11	11	$^{+8}_{-12}$
MILC	159	11	$^{+22}_{-9}$	$^{+21}_{-0}$

Table 2: f_{B_s} Estimates.

	MeV	stat	sys	quench
JLQCD	199	3	10	10
FNAL/UIUC	185	$^{+13}_{-8}$	9	$^{+10\%}_{-0\%}$
GLOK	175	8	13	10
MILC	175	10	$^{+28}_{-10}$	$^{+25}_{-1}$

Table 3: B_{B_d} Estimates.

	B_{B_d}	stat	sys
Gimenez and Martinelli	1.03	.06	.18
Christensen, Draper, McNeile, et.al.	1.40	.06	$^{+.04}_{-.26}$
Yamada, et. al.	1.17	.09	.05

this is not really at issue since ultimately we want to extract the bounds on the ρ, η plane. Recall that $V_{td} = A\lambda^3(1 - i\rho - \eta)$, so CKM matrix uncertainties in A and λ cancel to a large extent in the ratio.

$$\frac{|V_{td}|}{|V_{ts}|} = \lambda\sqrt{(1-\rho)^2 + \eta^2} \tag{30}$$

Since $\lambda = .22 \pm .004$, a 1.5% relative error, it does not contribute much to the uncertainty. So in the end, the annulus $\sqrt{(1-\rho)^2 + \eta^2}$ is diluted only by equation 29.

7 Bibliography

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