

Muon Cooling Channels

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My WWW home directory:

`http://keil.home.cern.ch/keil/
MuMu/Doc/Ring02/cooltalk.pdf`

Introduction

- Hierarchy of normalised emittances
 - $\varepsilon^i \gg \varepsilon^f$ for high merit factor
 - $\varepsilon^f \gg \varepsilon^{eq}$ for good efficiency
- Open or closed cooling channel
- Solenoid or dipole/quadrupole focusing
- Manipulate equations of Neuffer and of Wang and Kim
- Look for relations, conditions, and understanding, not for precision
- Average along channel
- Use results for assembling channel with magnets, RF cavities, absorbers, long before simulation
- Details in MUCOOL-257

Damping partition numbers

- Transverse partition number g_{\perp}

$$g_{\perp} = \left\langle 1 - D_y \frac{\frac{\partial}{\partial y} \left(\frac{dE}{ds} \right)}{\frac{dE}{ds}} \right\rangle$$

- $g_{\perp} = 1$ without wedges
- Longitudinal partition number g_{\parallel}

$$g_{\parallel} = \left\langle \frac{D_y \frac{\partial}{\partial y} \left(\frac{dE}{ds} \right)}{\frac{dE}{ds}} + \frac{\frac{\partial}{\partial \delta} \left(\frac{dE}{ds} \right)}{\frac{dE}{ds}} \right\rangle$$

- $\frac{\frac{\partial}{\partial \delta} \left(\frac{dE}{ds} \right)}{\frac{dE}{ds}} \approx -0.3$ for 200 MeV/c muons
- Sum rule

$$g_{\perp} + g_{\parallel} = 1 + \frac{\frac{\partial}{\partial \delta} \left(\frac{dE}{ds} \right)}{\frac{dE}{ds}}$$

Balbekov's RFOFO ring cooler

Definition of the longitudinal β -function β_s

- Linear longitudinal map \mathcal{M} for arc followed by RF station, operating on column vector $(ct, \delta p/p)^T$ with RF frequency f_{RF} , stable phase angle φ_s with origin at last zero crossing of RF voltage V :

$$\mathcal{M} = \begin{pmatrix} 1 & \frac{ch\eta}{f_{\text{RF}}\beta} \\ \frac{2\pi f_{\text{RF}}eV \cos \varphi_s}{Ec\beta} & 1 + \frac{2\pi\eta h eV \cos \varphi_s}{E\beta^2} \end{pmatrix}$$

- Find to lowest order in synchrotron tune Q_{\parallel} :

$$Q_{\parallel} = \sqrt{-\frac{\eta h eV \cos \varphi_s}{2\pi\beta^2 E}}$$

$$\beta_s = \frac{c}{f_{\text{RF}}} \sqrt{-\frac{\eta h E}{2\pi eV \cos \varphi_s}} = \sqrt{-\frac{c\eta C E}{2\pi\beta f_{\text{RF}} eV \cos \varphi_s}}$$

- β_s has dimension of length, as β_{\perp}

Damping rates and equilibrium emittances

- Wedge absorbers whose length depends on vertical coordinate y installed where $D_x = 0$ and $D_y \neq 0$, as in Balbekov's ring cooler
- Subscripts \perp and \parallel mark transverse and longitudinal motion, respectively.
- Find for cooling lengths and normalised equilibrium emittances:

$$s_{\perp}^{-1} = \frac{1}{pv} \left\langle \frac{dE}{ds} - \frac{D_y}{2} \frac{\partial}{\partial y} \left(\frac{dE}{ds} \right) \right\rangle$$

$$s_{\parallel}^{-1} = \frac{1}{pv} \left\langle D_y \frac{\partial}{\partial y} \left(\frac{dE}{ds} \right) + \frac{\partial}{\partial \delta} \left(\frac{dE}{ds} \right) \right\rangle$$

$$\varepsilon_{\perp}^{eq} = \frac{E\beta^3\gamma}{2} \left\langle \frac{\beta_{\perp} \left(\frac{13.6\text{MeV}}{pv} \right)^2 \frac{1}{X_0} + \mathcal{H}\langle\delta^2\rangle}{\frac{dE}{ds} - \frac{D_y}{2} \frac{\partial}{\partial y} \left(\frac{dE}{ds} \right)} \right\rangle$$

$$\varepsilon_{\parallel}^{eq} = \frac{E\beta^3\gamma}{2} \left\langle \frac{\beta_s \langle\delta^2\rangle + \gamma_{\parallel} D_y^2 \left(\frac{13.6\text{MeV}}{pv} \right)^2 \frac{1}{X_0}}{D_y \frac{\partial}{\partial y} \left(\frac{dE}{ds} \right) + \frac{\partial}{\partial \delta} \left(\frac{dE}{ds} \right)} \right\rangle$$

- s_{\perp}^{-1} and s_{\parallel}^{-1} have dimension of inverse length
- ε_{\perp} and ε_{\parallel} have dimension of length

Limits on amplitude functions

- Upper limit of the vertical amplitude function at the absorbers β_{\perp} due to the transverse equilibrium emittance

$$\beta_{\perp} \leq \frac{2\beta g_{\perp} E_{\mu} X_0 \varepsilon_{\perp}^{eq}}{(13.6\text{MeV})^2} \left(\frac{dE}{ds} \right)$$

- From the parameters of the absorber, we can calculate the mean square energy variation $d\langle\Delta E^2\rangle/ds$ due to straggling, and an upper limit for the longitudinal amplitude function β_s due to the longitudinal equilibrium emittance:

$$\beta_s \leq \frac{2E_{\mu} g_{\parallel} \varepsilon_{\parallel}^{eq}}{\beta} \left(\frac{\frac{dE}{ds}}{\frac{d\langle\Delta E^2\rangle}{ds}} \right)$$

- Fraction in the large brackets depends only on the absorber material
- β_{\perp} and β_s , and absorber length L_A essentially proportional to the products $g\varepsilon^{eq}$

Lower limit on vertical dispersion D_y

- Absolute value of $|D_y|$ at absorbers has lower limit due to wedge parameter, since it would be nice if absorber length were positive in whole vertical aperture. Taking an aperture radius equal to three RMS beam radii σ_y , and find with initial relative RMS momentum spread σ_e^i :

$$|D_y| \geq 3(1 - g_v)\sigma_y = 3(1 - g_v) \sqrt{\varepsilon_{\perp}^i \beta_{\perp} / (\beta\gamma) + (D_y \sigma_e^i)^2}$$

- Solving for $|D_y|$ yields:

$$|D_y| \geq 3(1 - g_v) \sqrt{\frac{\varepsilon_{\perp}^i \beta_{\perp}}{\beta\gamma[1 - (3(1 - g_v)\sigma_e^i)^2]}}$$

- $|D_y|$ is real and positive if $3(1 - g_v)\sigma_e^i \leq 1$. This condition is also found by assuming that σ_y is dominated by σ_e^i , and neglecting the contribution of the betatron oscillations.

Upper limits on vertical dispersion D_y

- Upper limits to $|D_y|$ due to cross-plane heating terms of equilibrium emittances
- $\mathcal{H} = D_y^2/\beta_\perp$ for absorber at waist with $\gamma_\perp = 1/\beta_\perp$ and $D'_y = 0$
- Heating term due to straggling \leq heating term due to multiple scattering, if

$$|D_y| \leq \frac{\beta_\perp (13.6\text{MeV}/pv)}{\sqrt{X_0 \langle \delta^2 \rangle}}$$

- Applying a similar argument to $\varepsilon_{\parallel}^{eq}$, heating term due to multiple scattering is \leq heating term due to energy straggling, if

$$|D_y| \leq \frac{\beta_s \sqrt{X_0 \langle \delta^2 \rangle}}{(13.6\text{MeV}/pv)}$$

- $\langle \delta^2 \rangle$ is rate of change per unit length of relative momentum error due to straggling
- Limit from second equation much higher
- When D_y is one half of limit, cross-plane heating term contributes one quarter of in-plane heating term to equilibrium emittance

Absorbers

- One kind of wedge-shaped, liquid hydrogen absorber with loss rate $dE/ds = 31.75 \text{ MeV/m}$ and radiation length $X_0 = 8.66 \text{ m}$
- Absorber occupancy factor is simply the ratio of $\langle dE/ds \rangle$ and dE/ds
- Length of absorber $L_A \leq 2\beta_{\perp}$
- Find spacing between neighbouring absorbers and period length L_P by dividing L_A by absorber occupancy factor
- Improve theory by replacing β_{\perp} by suitable average over length of absorber
- Arithmetic mean is $\langle \beta_{\perp} \rangle = 4\beta_{\perp}/3$
- Initial horizontal RMS beam radius σ_x and both divergences $\sigma_{x'}$ and $\sigma_{y'}$ at centre of absorber follow from initial transverse emittance ε_{\perp}^i and β_{\perp} , assuming that D_x , D'_x and D'_y vanish there:

$$\sigma_x = \sqrt{\frac{\varepsilon_{\perp}^i \beta_{\perp}}{\beta \gamma}} \qquad \sigma_{x'} = \sigma_{y'} = \sqrt{\frac{\varepsilon_{\perp}^i}{\beta \gamma \beta_{\perp}}}$$

RF System and Longitudinal Dynamics

- RF system must do four things:
 1. Compensate the energy loss in the absorbers
 2. Achieve the assumed initial longitudinal equilibrium emittance by having a satisfactory value of β_s
 3. Have a bucket height b_{RF} large enough to accept momentum spread in beam
 4. Have a bucket area A_{RF} that matches or is larger than $\varepsilon_{\parallel}^i$
- Average rate of acceleration in RF cavities equal to average rate of energy loss $\langle dE/ds \rangle$, if muon energies at entrance and exit of cooling channel are equal
- Average voltage gradient inside RF cavities is higher than average rate of acceleration for two reasons:
 - RF cavities occupy only a fraction of channel length
 - Peak voltage gradient is higher than accelerating gradient when muons are accelerated off the crest of RF wave form

Conditions on η and f_{RF} I

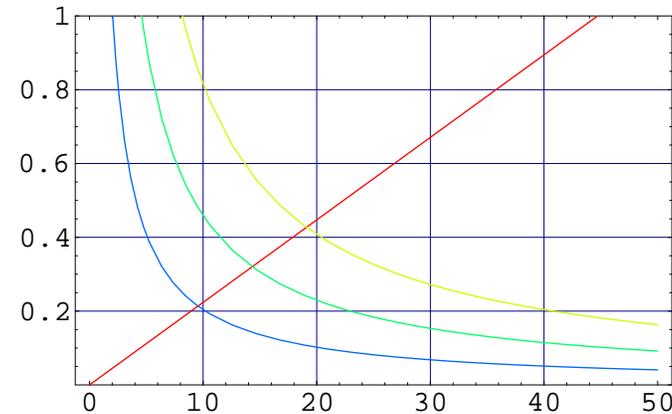
- Design value of β_s requires

$$\frac{|\eta|}{f_{\text{RF}}} \leq \frac{2\pi\beta\beta_s^2 \langle dE/ds \rangle}{cE \tan \varphi_s}$$

- Avoid fixing sign of η and quadrant of φ_s by taking absolute values where needed
- Bucket height $b_{\text{RF}} = 2\Delta p/p$ requires

$$|\eta| = \frac{Y^2(\varphi_s)}{\sin \varphi_s} \frac{\langle dE/ds \rangle c}{4\pi\beta E f_{\text{RF}}} \left(\frac{\Delta p}{p} \right)^{-2}$$

- $Y(\varphi) = \sqrt{|(\pi - 2\varphi) \sin \varphi - 2 \cos \varphi|}$ with $Y(0) = \sqrt{2}$ describes dependence of b_{RF} on φ



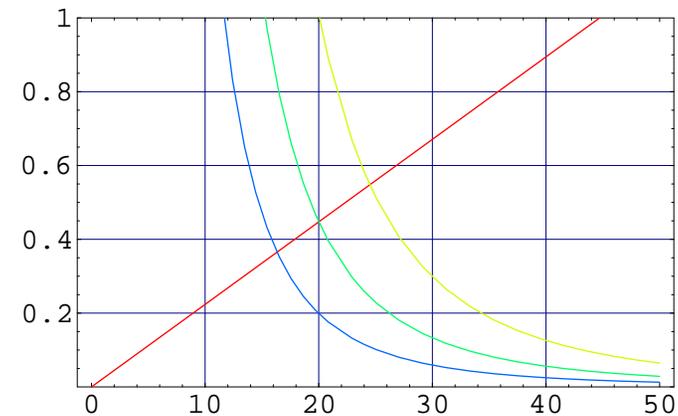
Upper limits for $|\eta|$ as function of f_{RF} in MHz at $\varphi_s = 2\pi/3$. Straight red curve shows limit due to β_s . Yellow, green and blue hyperbolic curves from above show limits for $\Delta p/p = 0.075, 0.1, 0.15$.

Conditions on η and f_{RF} II

- Bucket area $A_{\text{RF}} > 4\pi\varepsilon_{\parallel}^i$ requires

$$|\eta| \leq \frac{2\alpha^2(\varphi_s - \pi/2)\langle dE/ds \rangle \gamma}{(\pi\varepsilon_{\parallel}^i)^2 E_{\mu} f_{\text{RF}}^3 \sin \varphi_s} \left(\frac{\beta c}{\pi} \right)^3$$

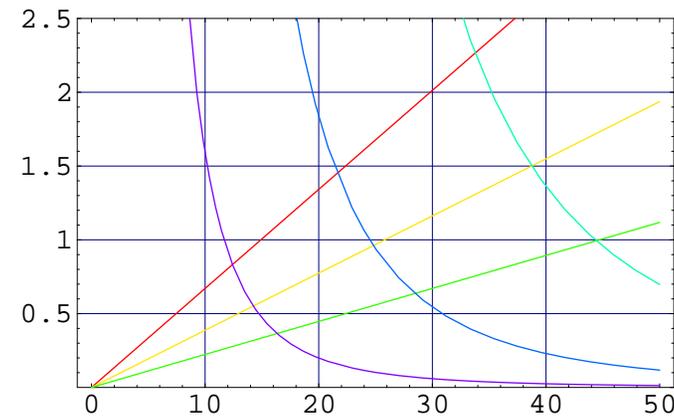
- Bucket area function $\alpha(\psi)$ with $\alpha(0) = 1$, $\alpha(\pi/2) = 0$ and ψ counted from crest
- Dôme gives power series for $\alpha(\psi)$



Upper limits for $|\eta|$ as function of f_{RF} in MHz at $\varphi_s = 2\pi/3$. Straight red curve shows limit due to β_s . Yellow, green and blue hyperbolic curves from above show the limits for $A_{\text{RF}} = 0.6\pi, 0.9\pi$ and 1.35π m.

Conditions on η and f_{RF} III

- Do not propose to operate muon cooling channel at limits of $|\eta|$ and f_{RF}
- Take them as starting point for scaling
- Doubling f_{RF} reduces $|\eta|$ by factor 8
- Maximum $|\eta|$ as function of f_{RF} given by β_s at low f_{RF} , by A_{RF} at high f_{RF} , by b_{RF} at intermediate f_{RF}
- Limit due to b_{RF} may be higher than the other two



Upper limits for $|\eta|$ as function of f_{RF} in MHz for $\varphi_s = 5\pi/6, 3\pi/4, 2\pi/3$ from above. Straight curves shows limit due to β_s . Hyperbolic curves show limits due to $A_{\text{RF}} = 1.35\pi$ m.

Conclusions

- To achieve assumed equilibrium emittances, β_{\perp} and β_s must be below upper limits
- $|D_y|$ must be within limits in order to achieve positive absorber length across vertical aperture and to avoid excessive cross-plane heating and associated increase in equilibrium emittances
- Liquid hydrogen absorbers must have length L_A , wedge parameter $D_y \ell' / \ell_0$, and spacing $\leq L_P$
- Practically feasible RF voltage gradients determine length of cooling channel
- η and f_{RF} must provide b_{RF} and A_{RF} matched to initial momentum spread and longitudinal emittance and β_s matched to $\varepsilon_{\parallel}^{eq}$
- For every factor of two in f_{RF} , $|\eta|$ goes down by a factor of eight
- At constant A_{RF} and φ_s , b_{RF} determines the accepted momentum range, and is proportional to f_{RF} .

Comparison with Kirk's Ring – Input Parameters

- Kirk's results are my input parameters and vice versa
- Swap horizontal and vertical because of wedge orientation

Parameter	Kirk	Program	
Design momentum p	250	250	MeV/c
Init. hor. emittance ε_x^i	26.55	26.55	mm
Init. vert. emittance ε_y^i	14.74	14.74	mm
Final vert. emittance ε_y^f	6.45	6.45	mm
Equil. vert. emittance ε_y^{eq}	4.47	4.47	mm
Init. long. emittance ε_z^i	14.92	14.92	mm
Final long. emittance ε_z^f	7.14	7.14	mm
Equil. long. emittance ε_z^{eq}	5.22	5.22	mm
Channel length	400	400	m

Comparison with Kirk's Ring – Output Parameters

Parameter	Kirk	Program	
$\Delta g_{\perp} = \frac{\partial}{\partial \delta} \left(\frac{dE}{ds} \right) / \left(\frac{dE}{ds} \right)$	-0.16	-0.159	
Vertical partition number	0.33	0.424	
Equil. hor. emittance ε_x^{eq}	1.01	1.89	mm
Final hor. emittance ε_x^f	2.11	2.40	mm
Long. partition number	0.51	0.417	
β_{\perp} at absorber	0.25	0.55	m
β_s at absorber	4.1	3.5	m
Min. D_y	0.46	0.135	m
Max. D_y	0.46	1.296	m
Wedge parameter Dl' / ℓ_0	0.642	0.553	
Merit factor M	31.8	40.8	

RF issues

- Earlier pages give corrected equations and new graphs
- Table below compares RF parameters
- Limit on $|\eta|$ and f_{RF} due to β_s and $A_{\text{RF}} \approx 4\pi\epsilon_x^i$
- Parameters from Kirk marked by $^+$

Parameter	Kirk	Program	
Stable phase angle φ_s	$\pi/6^+$	$5\pi/6^+$	rad
Relative RMS momentum spread $\Delta p/p$	0.0311^+	0.0311^+	
Slip factor $ \eta $	0.813^+	0.739	
RF frequency f_{RF}	200^+	216	MHz
Bucket area A_{RF}	0.201	0.188	m
Bucket height b_{RF}	0.0719	0.0855	
Full bucket length	788	727	mm
RMS bunch length σ_z	187^+	101	mm