Muon Cooling Channels

E. Keil

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My WWW home directory:

`http://keil.home.cern.ch/keil/`
`MuMu/Doc/Ring02/cooltalk.pdf`

Introduction

- Hierarchy of normalised emittances
  - $\varepsilon^i \gg \varepsilon^f$ for high merit factor
  - $\varepsilon^f \gg \varepsilon^{eq}$ for good efficiency
- Open or closed cooling channel
- Solenoid or dipole/quadrupole focusing
- Manipulate equations of Neuffer and of Wang and Kim
- Look for relations, conditions, and understanding, not for precision
- Average along channel
- Use results for assembling channel with magnets, RF cavities, absorbers, long before simulation
- Details in MUCOOL-257

Damping partition numbers

- Transverse partition number $g_\perp$

$$g_\perp = \left\langle 1 - D_y \frac{\frac{\partial}{\partial y}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)}{\frac{\mathrm{d}E}{\mathrm{d}s}} \right\rangle$$

- $g_\perp = 1$ without wedges
- Longitudinal partition number $g_{||}$

$$g_{||} = \left\langle \frac{D_y \frac{\partial}{\partial y}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)}{\frac{\mathrm{d}E}{\mathrm{d}s}} + \frac{\frac{\partial}{\partial \delta}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)}{\frac{\mathrm{d}E}{\mathrm{d}s}} \right\rangle$$

- $\frac{\frac{\partial}{\partial \delta}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)}{\frac{\mathrm{d}E}{\mathrm{d}s}} \approx -0.3$ for 200 MeV/c muons
- Sum rule

$$g_\perp + g_{||} = 1 + \frac{\frac{\partial}{\partial \delta}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)}{\frac{\mathrm{d}E}{\mathrm{d}s}}$$

Balbekov’s RFOFO ring cooler

Definition of the longitudinal β-function $\beta_{\|}$

- Linear longitudinal map $\mathcal{M}$ for arc followed by RF station, operating on column vector $(ct, \delta p/p)^T$ with RF frequency f_{RF}, stable phase angle φ_s with origin at last zero crossing of RF voltage V:

$$\mathcal{M} = \begin{pmatrix} 1 & \frac{ch\eta}{f_{\mathrm{RF}}\beta} \\ \frac{2\pi f_{\mathrm{RF}} eV \cos\varphi_s}{Ec\beta} & 1 + \frac{2\pi\eta heV\cos\varphi_s}{E\beta^2} \end{pmatrix}$$

- Find to lowest order in synchrotron tune $Q_{\|}$:

$$Q_{\|} = \sqrt{-\frac{\eta heV\cos\varphi_s}{2\pi\beta^2 E}}$$

$$\beta_{\|} = \frac{c}{f_{\mathrm{RF}}}\sqrt{-\frac{\eta hE}{2\pi eV\cos\varphi_s}} = \sqrt{-\frac{c\eta CE}{2\pi\beta f_{\mathrm{RF}} eV\cos\varphi_s}}$$

- $\beta_{\|}$ has dimension of length, as $\beta_{\perp}$

Damping rates and equilibrium emittances

- Wedge absorbers whose length depends on vertical coordinate y installed where $D_x = 0$ and $D_y \neq 0$, as in Balbekov's ring cooler
- Subscripts $\perp$ and $\|$ mark transverse and longitudinal motion, respectively.
- Find for cooling lengths and normalised equilibrium emittances:

$$
\begin{aligned}
s_{\perp}^{-1} &= \frac{1}{pv}\left\langle \frac{\mathrm{d}E}{\mathrm{d}s} - \frac{D_y}{2}\frac{\partial}{\partial y}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)\right\rangle \\
s_{\|}^{-1} &= \frac{1}{pv}\left\langle D_y\frac{\partial}{\partial y}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right) + \frac{\partial}{\partial \delta}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)\right\rangle \\
\varepsilon_{\perp}^{eq} &= \frac{E\beta^3\gamma}{2}\left\langle \frac{\beta_{\perp}\left(\frac{13.6\mathrm{MeV}}{pv}\right)^2\frac{1}{X_0} + \mathcal{H}\langle\delta^2\rangle}{\frac{\mathrm{d}E}{\mathrm{d}s} - \frac{D_y}{2}\frac{\partial}{\partial y}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)}\right\rangle \\
\varepsilon_{\|}^{eq} &= \frac{E\beta^3\gamma}{2}\left\langle \frac{\beta_{\|}\langle\delta^2\rangle + \gamma_{\|}D_y^2\left(\frac{13.6\mathrm{MeV}}{pv}\right)^2\frac{1}{X_0}}{D_y\frac{\partial}{\partial y}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right) + \frac{\partial}{\partial\delta}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)}\right\rangle
\end{aligned}
$$

- $s_{\perp}^{-1}$ and $s_{\|}^{-1}$ have dimension of inverse length
- $\varepsilon_{\perp}$ and $\varepsilon_{\|}$ have dimension of length

Limits on amplitude functions

- Upper limit of the vertical amplitude function at the absorbers $\beta_\perp$ due to the transverse equilibrium emittance

$$\beta_\perp \le \frac{2\beta g_\perp E_\mu X_0 \varepsilon_\perp^{eq}}{(13.6\mathrm{MeV})^2}\left(\frac{\mathrm{d}E}{\mathrm{d}s}\right)$$

- From the parameters of the absorber, we can calculate the mean square energy variation $\mathrm{d}\langle\Delta E^2\rangle/\mathrm{d}s$ due to straggling, and an upper limit for the longitudinal amplitude function $\beta_{||}$ due to the longitudinal equilibrium emittance:

$$\beta_{||} \le \frac{2E_\mu g_{||}\varepsilon_{||}^{eq}}{\beta}\left(\frac{\frac{\mathrm{d}E}{\mathrm{d}s}}{\frac{\mathrm{d}\langle\Delta E^2\rangle}{\mathrm{d}s}}\right)$$

- Fraction in the large brackets depends only on the absorber material
- $\beta_\perp$ and $\beta_{||}$, and absorber length L_A essentially proportional to the products $g\varepsilon^{eq}$

Lower limit on vertical dispersion D_y

- Absolute value of $|D_y|$ at absorbers has lower limit due to wedge parameter, since it would be nice if absorber length were positive in whole vertical aperture. Taking an aperture radius equal to three RMS beam radii σ_y, and find with initial relative RMS momentum spread σ_e^i:

$$|D_y| \geq 3(1-g_v)\sigma_y = 3(1-g_v)\sqrt{\varepsilon_\perp^i \beta_\perp/(\beta\gamma) + (D_y\sigma_e^i)^2}$$

- Solving for $|D_y|$ yields:

$$|D_y| \geq 3(1-g_v)\sqrt{\frac{\varepsilon_\perp^i \beta_\perp}{\beta\gamma[1-(3(1-g_v)\sigma_e^i)^2]}}$$

- $|D_y|$ is real and positive if $3(1-g_v)\sigma_e^i \leq 1$. This condition is also found by assuming that σ_y is dominated by σ_e^i, and neglecting the contribution of the betatron oscillations.

Upper limits on vertical dispersion D_y

- Upper limits to $|D_y|$ due to cross-plane heating terms of equilibrium emittances
- $\mathcal{H} = D_y^2/\beta_\perp$ for absorber at waist with $\gamma_\perp = 1/\beta_\perp$ and $D_y' = 0$
- Heating term due to straggling $\leq$ heating term due to multiple scattering, if

$$|D_y| \leq \frac{\beta_\perp (13.6\mathrm{MeV}/pv)}{\sqrt{X_0 \langle \delta^2 \rangle}}$$

- Applying a similar argument to $\varepsilon_\parallel^{eq}$, heating term due to multiple scattering is $\leq$ heating term due to energy straggling, if

$$|D_y| \leq \frac{\beta_\parallel \sqrt{X_0 \langle \delta^2 \rangle}}{(13.6\mathrm{MeV}/pv)}$$

- $\langle \delta^2 \rangle$ is rate of change per unit length of relative momentum error due to straggling
- Limit from second equation much higher
- When D_y is one half of limit, cross-plane heating term contributes one quarter of in-plane heating term to equilibrium emittance

Absorbers

- One kind of wedge-shaped, liquid hydrogen absorber with loss rate $\mathrm{d}E/\mathrm{d}s = 31.75$ MeV/m and radiation length $X_0 = 8.66$ m
- Absorber occupancy factor is simply the ratio of $\langle \mathrm{d}E/\mathrm{d}s \rangle$ and $\mathrm{d}E/\mathrm{d}s$
- Length of absorber $L_A \leq 2\beta_\perp$
- Find spacing between neighbouring absorbers and period length L_P by dividing L_A by absorber occupancy factor
- Improve theory by replacing $\beta_\perp$ by suitable average over length of absorber
- Arithmetic mean is $\langle \beta_\perp \rangle = 4\beta_\perp/3$
- Initial horizontal RMS beam radius σ_x and both divergences $\sigma_{x'}$ and $\sigma_{y'}$ at centre of absorber follow from initial transverse emittance $\varepsilon^i_\perp$ and $\beta_\perp$, assuming that D_x, D'_x and D'_y vanish there:

$$\sigma_x = \sqrt{\frac{\varepsilon^i_\perp \beta_\perp}{\beta\gamma}} \qquad \sigma_{x'} = \sigma_{y'} = \sqrt{\frac{\varepsilon^i_\perp}{\beta\gamma\beta_\perp}}$$

RF System and Longitudinal Dynamics

- RF system must do four things:
  1. Compensate the energy loss in the absorbers
  2. Achieve the assumed initial longitudinal equilibrium emittance by having a satisfactory value of $\beta_{||}$
  3. Have a bucket height b_{RF} large enough to accept momentum spread in beam
  4. Have a bucket area A_{RF} that matches or is larger than $\varepsilon_{||}^{i}$
- Average rate of acceleration in RF cavities equal to average rate of energy loss $\langle \mathrm{d}E/\mathrm{d}s \rangle$, if muon energies at entrance and exit of cooling channel are equal
- Average voltage gradient inside RF cavities is higher than average rate of acceleration for two reasons:
  - RF cavities occupy only a fraction of channel length
  - Peak voltage gradient is higher than accelerating gradient when muons are accelerated off the crest of RF wave form

Conditions on η and f_{RF} I

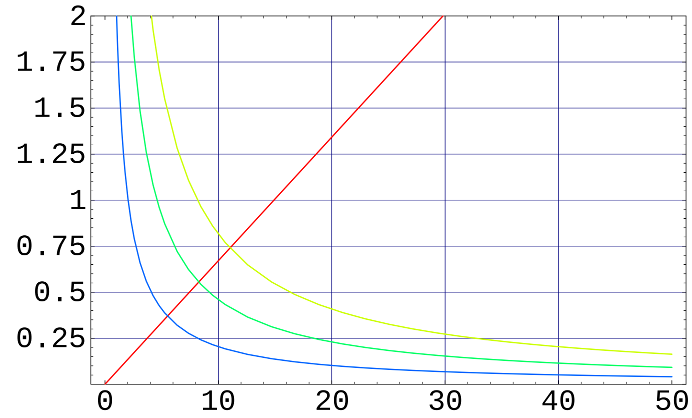
- Design value of $\beta_{||}$ requires

$$\frac{|\eta|}{f_{\mathrm{RF}}} \leq \frac{2\pi\beta\beta_{||}^2\langle\mathrm{d}E/\mathrm{d}s\rangle\tan\varphi_s}{cE}$$

- Avoid fixing sign of η and quadrant of φ_s by taking absolute values where needed
- Bucket height $b_{\mathrm{RF}} > \Delta p/p$ requires

$$|\eta| = \frac{Y^2(\varphi_s)}{\sin\varphi_s}\frac{\langle\mathrm{d}E/\mathrm{d}s\rangle c}{\pi\beta E f_{\mathrm{RF}}}\left(\frac{\Delta p}{p}\right)^{-2}$$

- $Y(\varphi) = \sqrt{|(\pi-2\varphi)\sin\varphi - 2\cos\varphi|}$ with $Y(0)=\sqrt{2}$ describes dependence of b_{RF} on φ



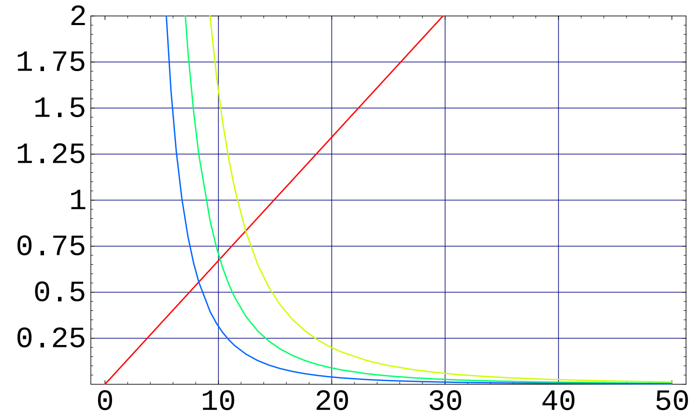
Upper limits for $|\eta|$ as function of f_{RF} in MHz. Straight red curve shows limit due to $\beta_{||}$. Yellow, green and blue hyperbolic curves from above show limits for $b_{\mathrm{RF}} = 0.15, 0.2, 0.3$ at $\varphi_s = 2\pi/3$.

Conditions on η and f_{RF} II

- Bucket area $A_{\mathrm{RF}} > 4\pi\varepsilon^i_{\parallel}$ requires

$$|\eta| \leq \frac{2\alpha^2(\varphi_s - \pi/2)\langle \mathrm{d}E/\mathrm{d}s\rangle\gamma}{(\pi\varepsilon^i_{\parallel})^2 E_\mu f^3_{\mathrm{RF}} \sin\varphi_s} \left(\frac{\beta c}{\pi}\right)^3$$

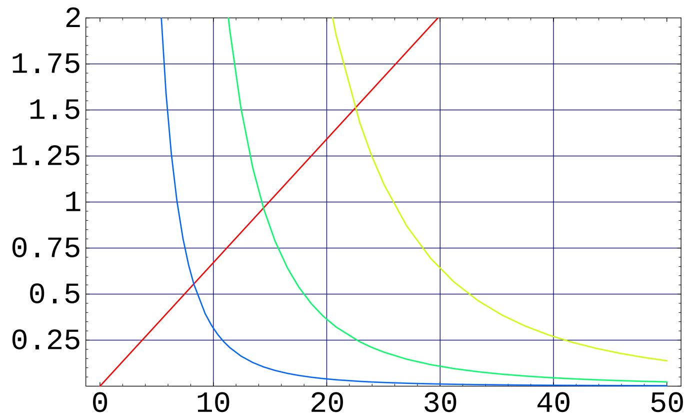
- Bucket area function $\alpha(\psi)$ with $\alpha(0) = 1$, $\alpha(\pi/2) = 0$ and ψ counted from crest
- Dôme gives power series for $\alpha(\psi)$



Upper limits for $|\eta|$ as function of f_{RF} in MHz. Straight red curve shows limit due to $\beta_{\parallel}$. Yellow, green and blue hyperbolic curves from above show the limits for $A_{\mathrm{RF}} = 0.6\pi, 0.9\pi$ and 1.35π m at $\varphi_s = 2\pi/3$.

Conditions on η and f_{RF} III

- Do not propose to operate muon cooling channel at limits of $|\eta|$ and f_{RF}
- Take them as starting point for scaling
- Doubling f_{RF} reduces $|\eta|$ by factor 8
- Maximum $|\eta|$ as function of f_{RF} given by $\beta_{||}$ at low f_{RF}, by A_{RF} at high f_{RF}, by b_{RF} at intermediate f_{RF}
- Limit due to b_{RF} may be higher than the other two



Upper limits for $|\eta|$ as function of f_{RF} in MHz. Straight red curve shows limit due to $\beta_{||}$. Yellow, green and blue hyperbolic curves from above show limits for $\varphi_s = 5\pi/6, 3\pi/4, 2\pi/3$ and $A_{\mathrm{RF}} = 1.35\pi$ m.

Conclusions

- To achieve assumed equilibrium emittances, $\beta_\perp$ and $\beta_{||}$ must be below upper limits
- $|D_y|$ must be within limits in order to achieve positive absorber length across vertical aperture and to avoid excessive cross-plane heating and associated increase in equilibrium emittances
- Liquid hydrogen absorbers must have length L_A, wedge parameter $D_y\ell'/\ell_0$, and spacing $\leq L_P$
- Practically feasible RF voltage gradients determine length of cooling channel
- η and f_{RF} must provide b_{RF} and A_{RF} matched to initial momentum spread and longitudinal emittance and $\beta_{||}$ matched to $\varepsilon_{||}^{eq}$
- For every factor of two in f_{RF}, $|\eta|$ goes down by a factor of eight
- At constant A_{RF} and φ_s, b_{RF} determines the accepted momentum range, and is proportional to f_{RF}.