Phase Ionization Cooling of a Muon Beam
1. General concepts

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1 Introduction

Table1 : Beam parameters after 6D cooling helical channel

Parameter	*Unit*	*equilibrium rms value*
Beam momentum, p	MeV/c	100
Synchrotron emittance, ε_s	μm	300
Relative momentum spread	%	2
Beam width due to $\Delta p / p$	mm	1.5
Bunch length	mm	11
Transverse emittances, $\varepsilon_+ / \varepsilon_-$	mm-mr	100/300
Beam widths, σ_1 / σ_2	mm	4.5/2.8

PIC combines the idea of a parametric resonance with the technique of ionization cooling to redistribute particles in phase space such that they become concentrated in a narrow phase angle.

In general, a parametric resonance is induced in an oscillating system by using a perturbing frequency that is the same as or a harmonic of a parameter of the system. Physicists are often first introduced to this phenomenon in the study of a rigid pendulum[i], where a periodic perturbation of the pivot point can lead to stable motion with the pendulum upside down. Half-integer resonant extraction from a synchrotron is another example familiar to accelerator physicists, where larger and larger radial excursions of particle orbits at successive turns are induced by properly placed quadrupole (and octupole) magnets that perturb the beam at a harmonic of the betatron frequency. In this case, the normal elliptical motion of a particle's horizontal coordinate in phase space at the extraction septum position becomes hyperbolic, $xx' = const$, leading to a beam emittance which has a wide spread in x and very narrow spread in x'.

In PIC, the same principle is used but the perturbation generates hyperbolic motion such that the emittance becomes narrow in x and wide in x' at certain positions as the beam passes down a line or circulates in a ring. Ionization cooling is then used to damp the angular spread of the beam. Figure 1 shows how the motion is altered by the perturbation. Figure 2 is schematic of the principle of ionization cooling showing how the angular divergence of the beam is damped. Figure 3 is a representation of the emittance evolution showing how the beam becomes concentrated in a narrow phase space area.

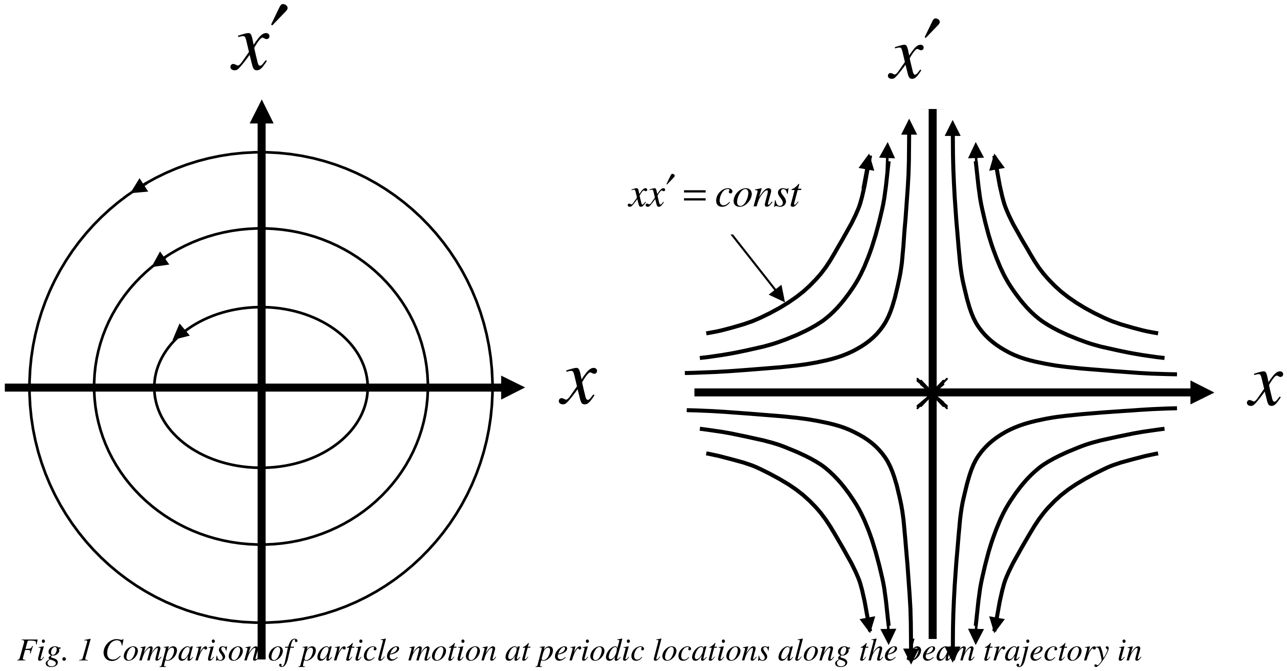

Fig. 1 Comparison of particle motion at periodic locations along the beam trajectory in transverse phase space for: LEFT ordinary oscillations and RIGHT hyperbolic motion induced by perturbations at a harmonic of the betatron frequency.

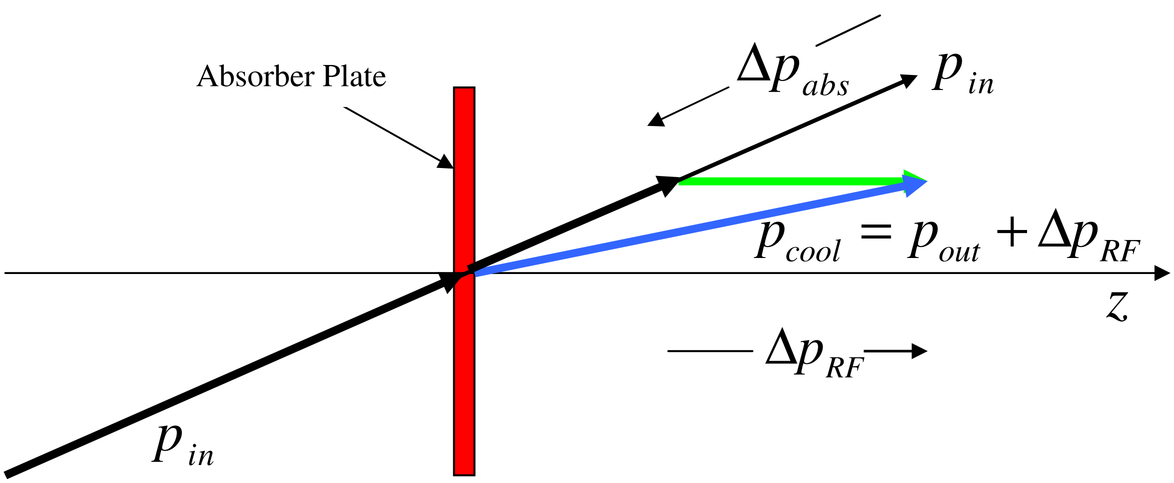

Fig. 2 Principle of transverse ionization cooling. A particle loses momentum in all three coordinates as it passes through an energy absorbing plate. Only the longitudinal component is replaced by RF fields, thereby reducing the angular divergence of the particle, x'.

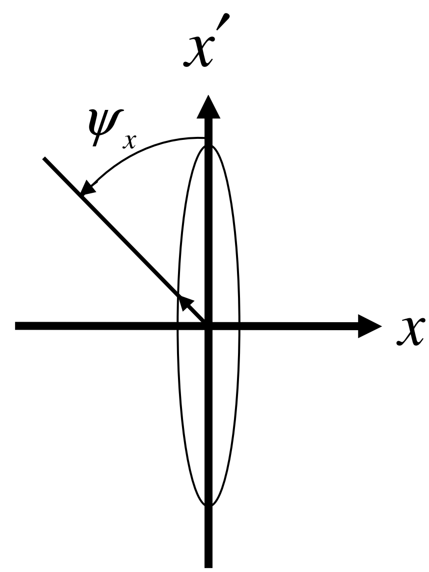

Fig. 3 Phase space compression. The spread in x *diminishes due to the parametric resonance motion while the spread in* x' *diminishes due to ionization cooling. The area of the phase space ellipse is reduced as the particles are restricted to a narrow range of phase angle,* ψ_x.

2 A principal PIC model

2.1 Hyperbolic dynamics and beam envelope

First, we introduce the basic principles of phase ionization cooling. Let there be a periodic focusing lattice of period λ along the beam path with coordinate z. As is well-known, particle tracking or mapping is based on a single period transformation matrix, M (between two selected points, z_0 and $z_0+\lambda$), for particle transverse coordinate and angle,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{z_0+\lambda} = M_x \begin{pmatrix} x \\ x' \end{pmatrix}_{z_0}$$

with a similar expression for the y coordinate.

The matrices M_x and M_y are symplectic or canonical, which means each has determinant equal to one. Otherwise, the matrix elements are arbitrary in general. Thus, each can be represented in a general form convenient for later discussions as follows:

$$M = \begin{pmatrix} e^{-\lambda\Lambda_d}\cos\psi & g\sin\psi \\ -\frac{1}{g}\sin\psi & e^{\lambda\Lambda_d}\cos\psi \end{pmatrix}$$

In particular, the optical period can be designed in a way that $\sin\psi = 0$, (i.e. $\psi = \pi$ or $\psi = 2\pi$), then the evolving particle coordinate and angle (or momentum) appear uncoupled:

$$\begin{aligned} (x)_{z_0+\lambda} &= \pm e^{-\lambda\Lambda_d}(x)_{z_0} \\ (x')_{z_0+\lambda} &= \pm e^{\lambda\Lambda_d}(x')_{z_0} \end{aligned}.$$

Thus, if the particle angle at point z_0 grows ($\Lambda_d > 0$), then the transverse position experiences damping, and vice versa. Liouville's theorem is not violated, but particle trajectories in phase space are hyperbolic ($xx' = const$); this is an example of a parametric resonance. Exactly between the two resonance focal points the opposite situation occurs where the transverse particle position grows from period to period, while the angle damps.

2.2 Phase cooling by using thin absorber plates

Stabilizing absorber effect

If we now introduce an energy absorber plate of thickness w at each of the resonance focal points as shown in figure 4, ionization cooling damps the angle spread with a rate Λ_c. Here we assume balanced 6D ionization cooling, where the three partial cooling decrements have been equalized using emittance exchange techniques as described in reference [iii]:

$$\Lambda_c = \frac{1}{3}\Lambda, \quad \Lambda = 2\frac{\langle \gamma'_{abs} \rangle}{\gamma} = 2\frac{\gamma'_{acc}}{\gamma}, \quad \langle \gamma'_{abs} \rangle = \gamma'_{abs} 2w/\lambda,$$

where γ'_{abs} and γ'_{acc} are the intrinsic absorber energy loss and the RF acceleration rate, respectively. If $\Lambda_d = \Lambda_c / 2$, then the angle spread and beam size are damped with decrement $\Lambda_c / 2$:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{z_0+\lambda} = e^{-\lambda\Lambda_c/2} \begin{pmatrix} x \\ x' \end{pmatrix}_{z_0}.$$

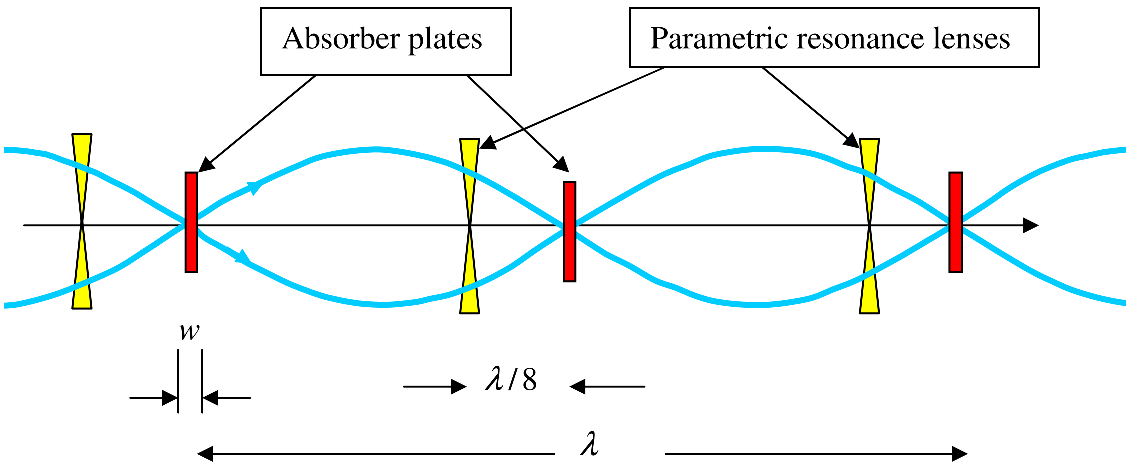

Fig. 4 Conceptual diagram of a beam cooling channel in which hyperbolic trajectories are generated in transverse phase space by perturbing the beam at the betatron frequency, a parameter of the beam oscillatory behavior. Neither the focusing magnets that generate the betatron oscillations nor the RF cavities that replace the energy lost in the absorbers are shown in the diagram.

Reduction of phase diffusion

The rms angular spread is increased by scattering and decreased by cooling,

$$\frac{d}{dz}\overline{(x')^2} = \frac{(Z+1)}{2\gamma\beta^2}\frac{m_e}{m_\mu}\Lambda - \Lambda_c \overline{(x')^2},$$

which leads to the equilibrium angular spread at the focal point:

$$\overline{(x')^2_{eq}} = \frac{1}{\Lambda_c}\frac{d}{dz}\overline{(x')^2_{z_0}} = \frac{3}{2}\frac{(Z+1)}{\gamma\beta^2}\frac{m_e}{m_\mu}.$$

The rms product $\left[\overline{(x^2)}\cdot\overline{(x')^2}\right]^{\frac{1}{2}}_{z_0}$ determines the effective 2D beam phase space volume, or emittance.

Taking into account the continuity of x in collisions, the diffusion rate of particle position at the focus is a function of $s = z - z_0$, the local position of the beam within the absorber:

$$\delta(x)_{z_0} = -s\delta x', \quad -\frac{w}{2} \le s \le \frac{w}{2}, \quad \frac{d}{dz}\overline{(\delta x)^2_{z_0}} = \frac{w^2}{12}\frac{d}{dz}\overline{(\delta x')^2}.$$

Reduction of the equilibrium emittance

Thus, in our cooling channel with resonance optics and correlated absorber plates, the equilibrium beam size at the plates is determined not by the characteristic focal parameter of the optics, $\lambda / 2\pi$, but by the thickness of absorber plates, w. Hence, the equilibrium emittance due to angle scattering is equal to

$$(\varepsilon_x)_{eq} = \gamma\beta\left[\overline{(x^2)}\cdot\overline{(x')^2}\right]^{\frac{1}{2}}_{z_0} = \gamma\beta\frac{w}{2\sqrt{3}}\overline{(x')^2_{z_0}} = \frac{\sqrt{3}}{4\beta}(Z+1)\frac{m_e}{m_\mu}w.$$

The emittance reduction by PIC is improved compared to a conventional cooling channel by a factor

$$\frac{\pi}{\sqrt{3}}\frac{w}{\lambda} = \frac{\pi}{2\sqrt{3}}\frac{\gamma'_{acc}}{\gamma'_{abs}}.$$

Using the well-known formula for the instantaneous energy loss rate in an absorber, we find an explicit expression for the transverse equilibrium emittance (normalized) due to angle scattering that can be achieved using PIC:

$$\varepsilon_x = \frac{\sqrt{3}}{16}\beta\left(1+\frac{1}{Z}\right)\frac{(\lambda/2\pi)}{nr_e^2 \log}\gamma'_{acc}.$$

Here Z and n are the absorber atomic number and concentration, m_μ the muon mass, r_e the classical electron radius, and β is the muon velocity. Here log is a symbol for the Coulomb logarithm of ionization energy loss for fast particles:

$$\log \equiv \ln\left(\frac{2p^2}{h\nu\, m_\mu}\right) - \beta^2,$$

with $h\nu$ the effective ionization potential[ii]. A typical magnitude of the log is about 12 for our conditions. The equilibrium emittance in the resonance channel is primarily determined by the absorber atomic concentration, and it decreases with beam energy in the non-relativistic region.

2.3 Compatibility of transverse phase cooling with emittance exchange

Emittance exchange by using wedge absorber plates

In order to prevent the energy spread growth in the beam due to the energy straggling in absorber, one has to use the wedge absorber plates and introduce the dispersion, i.e. make the beam orbit energy-dependent [ ].. Such dependence results from the beam bend by a dipole field (alternating along beam line). As usual, particle coordinate relatively a reference orbit can be represented as a superposition

$$x = D\frac{\Delta\gamma}{\gamma\beta^2} + x_b \qquad (\ )$$

at a condition that D and x_b do not interfere on particle trajectory. In contrary, the absorber wedge arrow (i.e. the gradient of the plate width) must alternate coherently with the D oscillation. Considering at first the effects of energy loss in wedge absorber plates, we find a systematic change of particle energy and position x_b at plates:

$$<\Delta\gamma'>=<\frac{\partial\gamma'}{\partial\gamma}>\Delta\gamma + <\frac{\partial\gamma'}{\partial x}x>=\frac{\Lambda}{2\beta^2}(\frac{2}{\gamma^2}-\frac{D_0}{h})\Delta\gamma;$$
$$<({x_{0b}}^2)'>=2(-\Lambda_d {x_{0b}}^2 + <x_{0b}\frac{\partial x_{0b}}{\partial\gamma}\Delta\gamma'>) = -2(\Lambda_d - \frac{\Lambda}{2\beta^2}\frac{D_0}{h}){x_{0b}}^2.$$

Here we introduced a parameter *h* as an effective height of the absorber wedge:

$$h^{-1} = \frac{1}{\gamma'}\frac{\partial\gamma'}{\partial x}$$

Thus, if the correlator Dh^{-1} is positive, we gain a damping of the energy spread while the phase cooling decrement decreases. Let us assume an arrangement that makes the decrements of the three emittances equal to Λ/3 yet leaving equal the damping decrements of beam size and angle spread at absorber plates. Then, we obtain the following relationships:

$$\frac{D}{h} = 2(1-\frac{2}{3}\beta^2), \qquad . \qquad (\ )$$
$$\frac{\Lambda_d}{\Lambda} = \frac{1}{2}(\frac{1}{\beta^2}-\frac{1}{3})$$

Transverse phase diffusion due to the energy straggling

The dispersion introduced for longitudinal cooling will also be responsible for excitation of transverse emittances because of straggling, i.e. stochastic change of particle energy at scattering in on electrons in absorber [ ]. The related change of particle 'free' coordinate x_b after scattering in a plate can be found simply taking again into account the continuity of the total coordinate x:

$$D\frac{\delta\gamma}{\gamma\beta^2}+\delta x_b=0.$$

$$\frac{d}{ds}<(\delta x_b)^2>=\frac{1}{2}<\frac{D^2}{\gamma^2\beta^4}\frac{d}{ds}(\delta\gamma)^2>=\frac{\Lambda}{8}\frac{\gamma^2+1}{\gamma\beta^2\log}\frac{m_e}{m_\mu}D^2$$

Taking into account relationship ( ), we obtain formula for the total diffusion rate of particle "betatron coordinate" at plates as follows:

$$({x_b}^2)'=\frac{\Lambda}{8\gamma\beta^2}\frac{m_e}{m_\mu}[\frac{Z+1}{3}w^2+\frac{\gamma^2+1}{\log}4h^2(1-\frac{2}{3}\beta^2)^2] \qquad (\ )$$

2.4 The equilibrium for a monochromatic beam

By combining the total phase diffusion from ( ) and ( ) and assuming the equalized cooling decrements as shown in ( ), we then obtain the equilibrium transverse size at plates and equilibrium emittance:

$$\sigma_b{}^2=\frac{1}{8\gamma\beta^2}\frac{m_e}{m_\mu}[(Z+1)w^2+12\frac{\gamma^2+1}{\log}h^2(1-\frac{2}{3}\beta^2)^2] \qquad (\ )$$

This expression to be valid requires $\sigma_b << h$. Let us introduce a ratio
$\xi=(\sigma_b/h)<<1$
and rewrite equation ( ) as

$$\sigma_b{}^2=\frac{Z+1}{8\gamma\beta^2}\frac{m_e}{m_\mu}w^2\frac{1}{1-\frac{3}{2\xi^2\beta^2}\frac{m_e}{m_\mu}\frac{\gamma^2+1}{\gamma\log}(1-\frac{2}{3}\beta^2)^2}$$

As one can see now, the straggling impact on transverse emittance is insignificant, if

$$\xi>>(1-\frac{2}{3}\beta^2)[\frac{3}{2\beta^2}\frac{\gamma^2+1}{\gamma\log}\frac{m_e}{m_\mu}]^{1/2},$$

at a reasonably small chosen ξ value. If this condition is satisfied, then we return to the expression for minimum emittance of the phase cooling as in ( ).

3. Longitudinal cooling

3.1 Basic equations

Considering the longitudinal cooling accompanying the resonance transverse cooling (PIC), we assume a periodic system of RF resonators installed in correlation with a periodic system of magnets that provide beam focusing required for PIC and dispersion for an optimum emittance exchange as described above. For certainty and simplicity sake, assume a plane curved beam orbit, though the plane of beam bend interchanges periodically between horizontal to vertical polarization for a symmetric emittance exchange. Then, there are the equations of longitudinal particle motion in terms of energy and RF phase deviation from the reference (equilibrium) particle, as follows:

$$\Delta\gamma' = -\mathrm{E}_1(z)\varphi - 2\gamma'_a \frac{\Delta\gamma}{\beta^2\gamma^3} + \frac{\partial\gamma'_a}{\partial x}x + \delta\gamma'_a$$
$$\varphi' \equiv \omega\tau' = \frac{\omega}{\beta}(Kx - \frac{\Delta\gamma}{\gamma^3\beta^2})$$

Here $\mathrm{E}(z)$ is effective RF electric field amplitude including an average but also alternating component, $\delta\gamma'_a$ is energy loss fluctuation (straggling), and *K* is the bend curvature. We assume that the beam bend at absorber plates and RF resonators sections is zero or negligible. To solve these equations, we use the expansion of the transverse coordinate according to equation ( ), then average the equations along the beam path. We will perform the averaging, accounting for the second order terms on alternating parts of $\mathrm{E}_1(z)$ and $\eta(z)$:

$$\mathrm{E}_1 = \overline{\mathrm{E}}_1 + \tilde{\mathrm{E}}_1(z); \qquad \eta \equiv \frac{1}{\gamma\beta^3}(KD - \frac{1}{\gamma^2}) = \overline{\eta} + \tilde{\eta} = \frac{1}{\gamma\beta^3}(\overline{KD} - \frac{1}{\gamma^2}) + \tilde{\eta}$$

Let us temporarily neglect the stochastic absorber effects in equations ( ). If, as usual, we replace the field amplitude E_1 (z) and compaction factor $\eta(z)$ by their average values, $\overline{\mathrm{E}}_1$ and $\overline{\eta}$, we then obtain a conventional type of equations of particle motion in RF field i.e. synchrotron oscillation in RF bucket:

$$\Delta\gamma' = -\overline{\mathrm{E}}_1\varphi$$
$$\varphi' = \omega\overline{\eta}\Delta\gamma,$$

To take into account the effects of the alternating parts, one has to distinguish between the systematic and periodic change of both variables, γ and φ:

$$\Delta\gamma = \Delta\overline{\gamma} + \tilde{\gamma}; \qquad \varphi = \omega\tau = \overline{\varphi} + \tilde{\varphi} = \omega(\overline{\tau} + \tilde{\tau}) \qquad ( )$$

In first approximation, the alternating parts of particle energy and phase can be found by integrating the alternating dynamical parts in equations ( ) and ( ):

$$\tilde{\gamma} \approx -\hat{\tilde{\mathrm{E}}}_1 \overline{\varphi}; \qquad \tilde{\varphi} \approx \omega(\hat{\tilde{\eta}}\Delta\overline{\gamma} - \frac{Dx' - D'x}{\beta}) . \qquad (\ )$$

Here we introduced the alternating integrals such that $\hat{\tilde{\mathrm{E}}}_1{}' = \tilde{\mathrm{E}}_1$, $\hat{\tilde{\eta}}' = \tilde{\eta}$. Let us consider now $\Delta\overline{\gamma}$ and $\overline{\tau}$ as new variables; substituting them to equations ( ) and ( ), we obtain equations for these variables as follows:

$$\Delta\overline{\gamma}' = -\overline{\mathrm{E}}_1 \omega \overline{\tau} + \Lambda_{dz}\Delta\overline{\gamma} - \frac{\Lambda}{\beta^2}(\frac{D}{2h} - \frac{1}{\gamma^2})\Delta\overline{\gamma} + \delta\gamma'$$

$$\overline{\tau}' = \overline{\eta}\Delta\overline{\gamma} - \Lambda_{dz}\overline{\tau} - \hat{\tilde{\eta}}\delta\gamma' + \frac{D}{\beta}\delta\vartheta_x{}'$$

Here we use a notation Λ_{dz} for the *longitudinal hyperbolic tune*

$$\Lambda_{hz} \equiv \omega\overline{\hat{\tilde{\mathrm{E}}}_1 \tilde{\eta}} = -\omega\overline{\tilde{\mathrm{E}}_1 \hat{\tilde{\eta}}} \,.$$

This term may play an important role when either $\overline{\eta}$ or both $\overline{\eta}$ and $\overline{\mathrm{E}}_1$ appear small or zero. By organizing interference between alternating of η and E_1, one can effectively introduce damping in phase motion and provide phase cooling. There also are two stochastic terms in equation for phase $\overline{\tau}$ caused by transverse-longitudinal coupling. Considering that the longitudinal cooling is performed in parallel to the transverse phase cooling at condition ( ) and assuming $\Lambda_{dz} = \Lambda / 6$, we obtain the following final equations of the longitudinal cooling:

$$\Delta\overline{\gamma}' + \omega\overline{\mathrm{E}}_1\overline{\tau} = -\frac{\Lambda}{6}\Delta\overline{\gamma} + \delta\gamma'$$

$$\overline{\tau}' - \overline{\eta}\Delta\overline{\gamma} = -\frac{\Lambda}{6}\overline{\tau} - \hat{\tilde{\eta}}\delta\gamma' + \frac{2h}{\beta}(1 - \frac{2}{3}\beta^2)\delta\vartheta'$$

Based on these equations, the equilibrium solution for energy spread and bunch length in general case is found and analyzed in Appendix. Below, we find equilibrium in two characteristic situations: cooling along isochronous line (i.e. $\overline{\eta} = 0$) at $\overline{\mathrm{E}}_1 = 0$, and cooling in RF bucket.

3.2 Hyperbolic longitudinal cooling

$$\frac{d}{dz}(\Delta\bar{\gamma})^2 = -\frac{\Lambda}{3}[(\Delta\bar{\gamma})^2 - (\Delta\bar{\gamma})_0{}^2]$$
$$\frac{d}{dz}\bar{\tau}^2 = -\frac{\Lambda}{3}(\bar{\tau}^2 - \bar{\tau}_0{}^2)$$
$$(\Delta\bar{\gamma})^2{}_{eq} = (\Delta\bar{\gamma})_0{}^2 \equiv \frac{3m_e}{4m_\mu}\frac{\gamma^2+1}{\log}\gamma\beta^2$$
$$\bar{\tau}^2{}_{eq} = \bar{\tau}^2{}_0 \equiv (\Delta\bar{\gamma})_0{}^2\hat{\bar{\eta}}_a{}^2 + 6\frac{m_e}{m_\mu}\frac{Z+1}{\gamma\beta^4}h^2(1-\frac{2}{3}\beta^2)^2],$$

and

$$(\varepsilon_z)_{eq} = \sqrt{(\Delta\bar{\gamma})_0{}^2\,\bar{\tau}_0{}^2}$$

3.3 Longitudinal cooling in RF bucket

$$I_s(\Delta\bar{\gamma},\bar{\tau}) = \frac{1}{2}[\sqrt{\frac{\bar{\eta}}{\omega\bar{\mathrm{E}}_1}}(\Delta\bar{\gamma})^2 + \sqrt{\frac{\omega\bar{\mathrm{E}}_1}{\bar{\eta}}}\bar{\tau}^2]$$

$$\varepsilon_z{}' = -\frac{\Lambda}{3}[\varepsilon_z - (\varepsilon_z)_{eq}]$$
$$(\varepsilon_z)_{eq} = \frac{1}{2}[\sqrt{\frac{\bar{\eta}}{\omega\bar{\mathrm{E}}_1}}(\Delta\bar{\gamma})_0{}^2 + \sqrt{\frac{\omega\bar{\mathrm{E}}_1}{\bar{\eta}}}\bar{\tau}_0{}^2]$$

Since two components of adiabatic invariant ( ) are equal in average, we then obtain solutions for equilibrium energy spread and bunch length:

$$(\Delta\bar{\gamma})^2{}_{eq} = \frac{1}{2}[(\Delta\gamma)_0{}^2 + \frac{\omega\bar{\mathrm{E}}_1}{\bar{\eta}}\bar{\tau}_0{}^2], \qquad \bar{\tau}^2{}_{eq} = \frac{1}{2}[\bar{\tau}_0{}^2 + \frac{\bar{\eta}}{\omega\bar{\mathrm{E}}_1}(\Delta\bar{\gamma})_0{}^2]$$

Considering the equilibrium emittance as function of parameter $\frac{\bar{\eta}}{\omega\bar{\mathrm{E}}_1}$, we find that it takes minimum value at

$$\frac{\bar{\eta}}{\omega\bar{\mathrm{E}}_1} = \frac{\bar{\tau}_0{}^2}{(\Delta\bar{\gamma})_0{}^2}, \qquad ( )$$

then:

$$(\varepsilon_z)_{eq} = (\Delta\bar{\gamma})_0\,\bar{\tau}_0, \qquad (\Delta\bar{\gamma})_{eq} = (\Delta\bar{\gamma})_0, \qquad \bar{\tau}_{eq} = \bar{\tau}_0.$$

Phase Ionization Cooling
2. Parametric resonance regime of ionization cooling

In the consideration above we assumed the diagonal form of 2D transformation matrix between the absorber plates as shown in equation (   ). This assumption leads immediately to the hyperbolic beam evolution at absorber plates, reduction of phase diffusion and equilibrium emittance. Now, we will consider a resonance method to realize the hyperbolic dynamics in one of two planes of a focusing channel.

1 Parametric resonance in transverse beam motion

Parametric resonance at a homogeneous focusing

Let there be a focusing force in *x* plane the characteristic of which is composed of a constant part corresponding to oscillation frequency k_x and a small alternating part oscillating along beam path with a frequency *2k* :

$$x'' + k_x^{\ 2}(1 - 2\zeta \sin 2kz)x = 0,$$

where ζ = *const* «1 is the frequency modulation parameter. Let us represent the oscillator motion in terms of "slow" variables $a(z), b(z)$ which would be constant at $\zeta = 0$:

$$x = a\cos kz + b\sin kz,$$
$$x' = -ak\sin kz + bk\cos kz.$$

Apparently, *a* and *kb* represent particle coordinate and angle in x-plane, respectively, at points of the beam orbit where $\sin kz = 0$, while at points $\cos kz = 0$ the coordinate and angle are correspondently represented by *b* and *ka*. Resolving these equations on the variables *a* and *b*:

$$a = x\cos kz - (x'/k\ \ )\sin kz,$$
$$b = x\sin kz + (x'/k)\cos kz,$$

we find the derivatives a' and b', by taking into account the equation of motion for $x(z)$:

$$a' = -2k(\zeta \sin 2kz - \nu)x\sin kz \equiv -2k(\zeta \sin 2kz - \nu)(a\cos kz + b\sin kz)\sin kz,$$
$$b' = 2k(\zeta \sin 2kz - \nu)x\cos kz \equiv 2k(\zeta \sin 2kz - \nu)(a\cos kz + b\sin kz)\cos kz,$$

Here we introduced a notation for frequency detune, ν :

$$2\nu = (k_x / k)^2 - 1 .$$

Thus, we derived the equations of motion in terms of slow variables *a* and *b*. The coefficients of these equations are periodic with a period π / k . We assume now that both the parameters, ζ and ν, are small, then we can approximate change of *a* and *b* during a single period by simple integrating the equations over one period at *a =const, b = const* on the right, then we find:

$$\Delta a = -(\pi / 2)(\zeta a - 2\nu b); \Delta b = (\pi / 2)(\zeta b - 2\nu a),$$

or, in terms of the effective differential equations,

$$a' \equiv (k / \pi)\Delta a = -(\zeta / 2)ka + \nu kb,$$
$$b' \equiv (k / \pi)\Delta b = (\zeta / 2)kb - \nu ka.$$

These equations can also be obtained by an immediate averaging the equations ( ) on *z*.

Assume now for certainty $\zeta \rangle 0$, and let us observe the oscillator behavior dependence on detune ν . At $\nu = 0$, the beam experiences an exponentially increasing focusing at the points $\sin kz = 0$, i.e. at the middle between two successive maximum and minimum of the focusing strength (see figure   ), in other words – at the points where the focusing strength is equal to its average magnitude, yet decreases, or at points of phase advance 45° after the points of maximum focusing strength. Correspondently, the beam size grows while angle spread damps at points $\cos kz = 0$. Thus, at the exact resonance we observe the hyperbolic dynamics identical to that introduced in part 2.1.

Parametric resonance at a lumping focusing

The hyperbolic dynamic regime, actually, does not require for its realization a homogeneous focusing but can be arranged, as well, in a periodic lattice by implicating the parametric resonance technique. …

More text and formulas

4D parametric resonance in an axi-symmetrical focusing channel

More text and formulas

2 Basic equations of resonance cooling

Resonance equations with 6D cooling and stochastic force

More text and formulas

Detuning factors:

- *Angle aberration*
- *Non-linearity*
- *Chromaticity*
- *Energy-transverse emittance correlation in RF bucket*

More text

The constant and alternating components of the beta-function spreads

$$\Delta\beta = \Delta\overline{\beta} + \tilde{\beta}$$

- $\Delta\overline{\beta}$ *is constant function of the adiabatic invariants*
- $\tilde{\beta}$ *alternates with periods of the:*
  1) focusing lattice;
  2) energy oscillation in the RF bucket (if chromaticity is not compensated)

More text

- ***Requirement to the constant beta spreads***

$$\frac{\Delta\overline{\beta}}{\beta} << \frac{\Lambda}{3}\beta$$

More text

- ***Equilibrium phase aberration due to*** $\Delta\overline{\beta}$ ***:***

$$\Delta\overline{\psi} \approx \frac{\Delta\overline{\beta}}{\beta}\frac{3}{\Lambda\beta}$$

Could it be compensated after PIC?

More text

- ***Effect of the alternating parts***

$$\Delta\psi = \tilde{\psi}; \qquad \tilde{\psi}' = \tilde{\beta}$$

More text

3 Phase damping and beam equilibrium

4 Detuning compensation designs

Versions under study:

A) **Alternating solenoid channel**
   - Low bend and dispersion for emittance exchange
   - No compensation for chromaticity
   - Use energy-transverse emittance correlation in RF bucket
   - Use solenoid fringe non-linear field to compensate for angle aberration detune

Strong phase shrinkage is estimated

B) **Solenoid interchange with chicanes**
   - **Chromaticity compensated**
   - **Sextupoles in chicanes to compensate for angle aberration and non-linearity**
   - **Low dispersion in solenoids for emittance exchange**

Better phase shrinkage can be expected?

C) **Continuous helical channel**
   - Solenoid
   - Helical dipole + quad +sextupole + octupole (compensate for everything…)
   - Make two helical tunes equal (or ratio ½)
   - Use wedge absorber

[i] http://monet.physik.unibas.ch/~elmer/pendulum/parres.htm

[ii] L. D. Landau and E. M. Lifshits, Theoretical Physics v. 8, Electrodynamics of continuous media 1960 QC 518.L23.