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Coordinate Transformations of Harmonic Field Components

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1 Field representation

In the straight section of a magnet, where translational symmetry applies and the longitudinal component of the magnetic field vanishes, the magnetic field in the bore can be represented in terms of multipole coefficients defined according to the following expression:

$$\begin{aligned} B_\theta(r,\theta) &= \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}(B_n\cos n\theta - A_n\sin n\theta) \\ B_r(r,\theta) &= \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}(A_n\cos n\theta + B_n\sin n\theta) \end{aligned}$$

The same multipole coefficients appear in the equivalent expansion in complex notation:

$$B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty}(B_n + iA_n)\left(\frac{x+iy}{r_0}\right)^{n-1} \qquad (1)$$

It is also customary to express the harmonics in "units" of 10^{-4} of the main field component:

$$b_n = B_n/B_{main}\cdot 10^4 \qquad a_n = A_n/B_{main}\cdot 10^4$$

2 Rotations

Two polar coordinate systems $R(r,\theta_R)$ and $S(r,\theta_S)$ are considered. As shown in Figure 1, the two systems have the same origin and radial coordinate, while the relation between the azimuthal coordinates is given by a rotation $\theta_R = \theta_S + n\pi/2$. Note that *if the physical system is rotated rather than the reference frame*, the angles should be taken with the opposite sign i.e. the results for $\pi/2$ and $3\pi/2$ are exchanged.

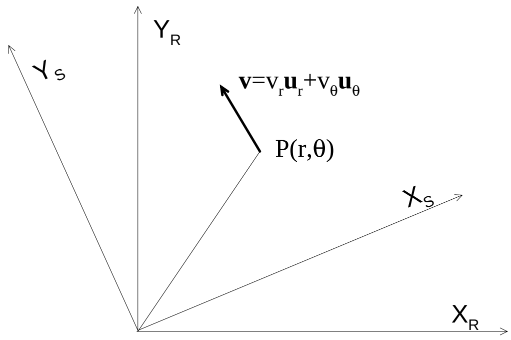

Figure 1: Reference frame definitions

Since the unit vectors at a given point P are the same for the two systems, the magnetic field components at the point P must be the same:

$$B_\theta^R(r,\theta_R) = B_\theta^S(r,\theta_S) \tag{2}$$
$$B_r^R(r,\theta_R) = B_r^S(r,\theta_S) \tag{3}$$

Following equation 1, the azimuthal component of the magnetic field can be expressed in the two frames in the following way:

$$B_\theta^R(r,\theta_R) = \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}(B_n^R\cos n\theta_R - A_n^R\sin n\theta_R) \tag{4}$$
$$B_\theta^S(r,\theta_S) = \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}(B_n^S\cos n\theta_S - A_n^S\sin n\theta_S) \tag{5}$$

Supposing that the multipole coefficients in the R frame are known, we want to find the values of the multipole coefficients as seen from the reference frame S, for the the following cases:

1. X_S, Y_S are rotated by an angle $\pi/2$ with respect to X_R, Y_R: $\theta_R = \theta_S + \frac{\pi}{2}$. From equation 4:

$$B_\theta^R(r,\theta_S+\tfrac{\pi}{2}) = \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}[B_n^R\cos(n\theta_S+n\tfrac{\pi}{2}) - A_n^R\sin(n\theta_S+n\tfrac{\pi}{2})] =$$

$$= \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}[B_n^R(\cos n\theta_S\cos n\tfrac{\pi}{2} - \sin n\theta_S\sin n\tfrac{\pi}{2}) - A_n^R(\sin n\theta_S\cos n\tfrac{\pi}{2} + \cos n\theta_S\sin n\tfrac{\pi}{2})]$$

It is convenient to separate two cases depending on wether n is odd or even:

- For $n = 2k - 1$ we have $\cos n\frac{\pi}{2} = 0$, while $\sin n\frac{\pi}{2} = (-1)^{k+1}$.

$$B_\theta^R(r, \theta_S + \tfrac{\pi}{2}) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left[-A_n^R(-1)^{k+1} \cos n\theta_S + B_n^R(-1)^{k+2} \sin n\theta_S\right]$$

  By comparison with equation 5:

$$B_n^S = (-1)^k A_n^R \qquad A_n^S = (-1)^{k+1} B_n^R$$

- For $n = 2k$ we have $\cos n\frac{\pi}{2} = (-1)^k$, while $\sin n\frac{\pi}{2} = 0$.

$$B_\theta^R(r, \theta_S + \tfrac{\pi}{2}) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left[(-1)^k B_n^R \cos n\theta_S - (-1)^k A_n^R \sin n\theta_S\right]$$

  By comparison with equation 5:

$$B_n^S = (-1)^k B_n^R \qquad A_n^S = (-1)^k A_n^R$$

2. X_S, Y_S are rotated by an angle π with respect to X_R, Y_R: $\theta_R = \theta_S + \pi$. This corresponds to a rotation of $\pi/2$ applied for 2 times:

   - For $n = 2k - 1$:

$$B_n^\pi = (-1)^k A_n^{\pi/2} = -B_n^0$$
$$A_n^\pi = (-1)^{k+1} B_n^{\pi/2} = -A_n^0$$

   - For $n = 2k$:

$$B_n^\pi = (-1)^k B_n^{\pi/2} = B_n^0$$
$$A_n^\pi = (-1)^k A_n^{\pi/2} = A_n^0$$

3. X_S, Y_S are rotated by an angle $3\pi/2$ with respect to X_R, Y_R: $\theta_R = \theta_S + 3\pi/2$. This corresponds to a rotation of $\pi/2$ plus a rotation of π:

   - For $n = 2k - 1$:

$$B_n^{3\pi/2} = (-1)^k A_n^\pi = (-1)^{k+1} A_n^0$$
$$A_n^{3\pi/2} = (-1)^{k+1} B_n^\pi = (-1)^k B_n^0$$

   - For $n = 2k$:

$$B_n^{3\pi/2} = (-1)^k B_n^\pi = (-1)^k B_n^0$$
$$A_n^{3\pi/2} = (-1)^k A_n^\pi = (-1)^k A_n^0$$

3 Reflection on the X-axis

If S is a reflection of R with respect to X_R ($X_S = X_R, Y_S = -Y_R$, $\theta_R = -\theta_S$), the relation between the unit vectors at the point P is: $\vec{u}_r^R(r, \theta_R) = \vec{u}_r^S(r, \theta_S)$, $\vec{u}_\theta^R(r, \theta_R) = -\vec{u}_\theta^S(r, \theta_S)$. From equation 4:

$$B_\theta^R(r, -\theta_S) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} [B_n^R \cos(-n\theta_S) - A_n^R \sin(-n\theta_S)] =$$

$$= \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} [B_n^R \cos n\theta_S + A_n^R \sin n\theta_S]$$

By comparison with equation 5, and considering that $B_\theta^R(r, -\theta_S) = -B_\theta^S(r, -\theta_S)$, one obtains:

$$B_n^S = -B_n^R \qquad A_n^S = A_n^R$$

It should be noted that in order to maintain a right-handed coordinate system, this transformation *must change the direction of the Z unit vector as well*: $Z_S = -Z_R$. The direction of the Z axis is related to the sign of the axial currents which generate the field. Since the different reference frames describe the same physical system (as implied by the condition $\vec{B}^R(P) = \vec{B}^S(P)$ at all points), *the generating currents in S will have the opposite sign with respect to that of the corresponding sources in R.*

4 Summary table

The equations relating the harmonic field components before and after a rotation or reflection of the reference frame are summarized in Table 1. For what concerns the *multipole coefficients expressed in "units"*, it should be noted that for those transformations in which the main field component changes sign, all the b'_n, a'_n coefficients undergo a change of sign with respect to the corresponding B'_n, A'_n.

Table 1: Harmonic field components after a coordinate transformation. B'_n, A'_n indicate the multipole components computed in the new coordinate system. B_n, A_n indicate the multipole components computed in the old coordinate system, which are supposed to be known.

Coordinate transformation	$n = 2k-1$		$n = 2k$	
	B'_n	A'_n	B'_n	A'_n
Rot. 0	B_n	A_n	B_n	A_n
Rot. $\pi/2$	$(-1)^k A_n$	$(-1)^{k+1} B_n$	$(-1)^k B_n$	$(-1)^k A_n$
Rot. π	$-B_n$	$-A_n$	B_n	A_n
Rot. $3\pi/2$	$(-1)^{k+1} A_n$	$(-1)^k B_n$	$(-1)^k B_n$	$(-1)^k A_n$
Refl. X	$-B_n$	A_n	$-B_n$	A_n