# Dimensional Regularization
# meets
# Freshman E&M

References:


M. Hans, Am.J.Phys. 51 (8), August (1983), p.694
C. Kaufman, Am.J.Phys. 37 (5), May (1969)



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CTEQ Summer School
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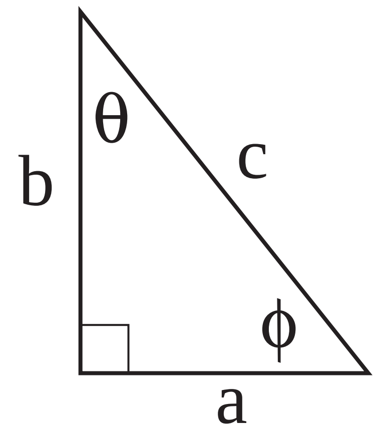
# Dimensional Analysis: The Pythagorean Theorem

**Goal:** Demonstrate $a^2 + b^2 = c^2$

**Method:** Dimensional Analysis:

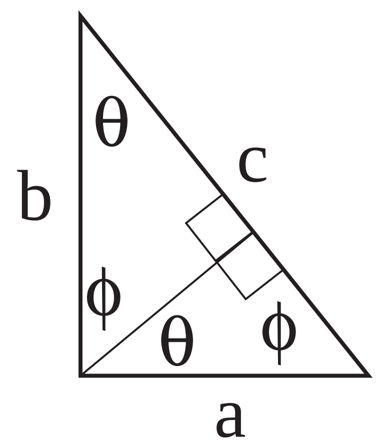
Total Area:

$$A_c = c^2 f(\theta,\phi)$$




Total Area

$$A_a + A_b = a^2 f(\theta,\phi) + b^2 f(\theta,\phi)$$

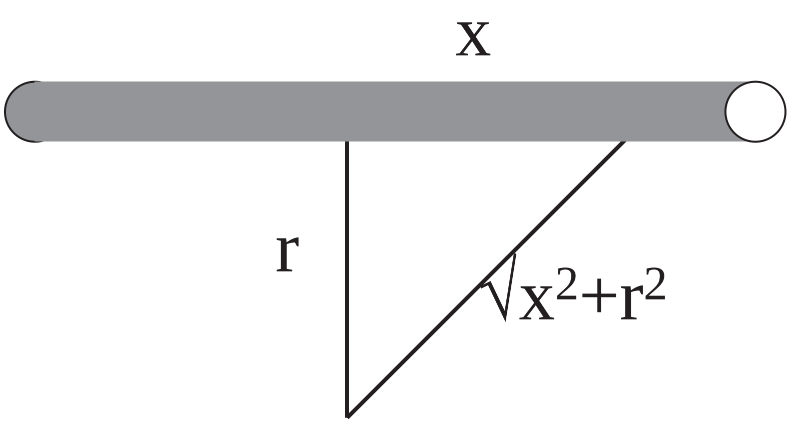



Total Area = Total Area

$$c^2 f(\theta,\phi) = a^2 f(\theta,\phi) + b^2 f(\theta,\phi)$$

$$\boxed{c^2 = a^2 + b^2}$$

# Potential due to Infinite Line of Charge




$$V[r] = \frac{\lambda}{4\pi\varepsilon_0}\int_{-} \frac{1}{\sqrt{x^2 + r^2}}\, dx =$$

with $\lambda = Q/x$ and $dQ = \lambda\, dx$

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## Scale Invariance:

V(r) Function of <u>only</u> $\lambda$ and r

$$V[k\,r] = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2 + (kr)^2}} = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{dy}{\sqrt{y^2 + r^2}} = V[r]$$

Implies: $V(r_1) = V(r_2)$ *after rescaling: x=ky*

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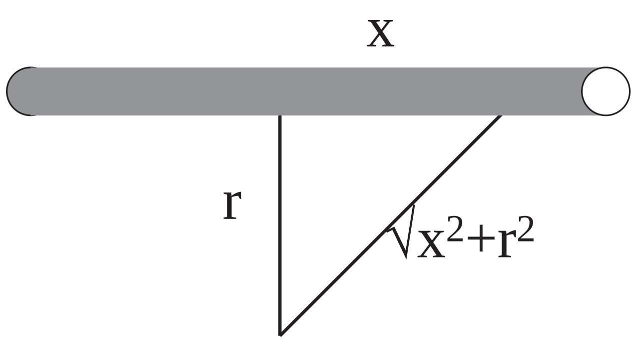
## Choices:

- $V(r) = c$ — a finite constant
- $V(r) = \infty$ — infinite
- $V(r) = V(r/\Lambda)$ — $\exists$ another length scale (Scale invariance is broken)

# Cutoff Regularization Method




Use L as cutoff

$$V[r] = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{1}{\sqrt{x^2+r^2}}\,dx = \frac{\lambda}{4\pi\varepsilon_0}\,\mathrm{Log}\left[\frac{L+\sqrt{L^2+r^2}}{-L+\sqrt{L^2+r^2}}\right]$$

Problem:

- V[r] depends on artificial regulator L
- We cannot remove the regulator L:

$$V[r] \xrightarrow[L\to\infty]{}$$

*All physical quantities are independent of the regulator!!!*

Electric Field:

$$E = \frac{-\partial V[r]}{\partial r} = \frac{\lambda}{2\pi\varepsilon_0\, r}\,\frac{L}{\sqrt{L^2+r^2}} \xrightarrow[L\to\infty]{} \frac{\lambda}{2\pi\varepsilon_0\, r}$$

Energy ~ δV:

$$\delta V = V[r_1] - V[r_2] = \frac{\lambda}{4\pi\varepsilon_0}\,\mathrm{Log}\left[\frac{r_2^2}{r_1^2}\right]$$

Problem solved at the expense of an extra scale, L

*BUT, we have a broken symmetry--translational invariance!!!*

# Dimensional Regularization Method

Idea: Compute V[r] in n-dimensions

Use: $d^n x = d\Omega_n \; x^{n-1} \; dx$

$d\Omega_n = 2\,\pi^{n/2} \,/\, \Gamma(n/2)$

| n | $d\Omega(n)$ |
|---|---|
| 1 | 2 |
| 2 | $2\,\pi$ |
| 3 | $4\,\pi$ |
| 4 | $2\,\pi^2$ |

Must introduce $\mu^{n-1}$ to ensure V[r] has correct dimension

$$V[r] = \frac{\lambda}{4\,\pi\varepsilon_0} \int_0 d\Omega[n] \, \frac{x^{n-1}}{\mu^{n-1}} \, \frac{dx}{\sqrt{x^2 + r^2}}$$

Result: with $n=1-2\varepsilon$

$$V[r] = \frac{\lambda}{4\,\pi\varepsilon_0} \, \frac{\Gamma\left[\frac{1-n}{2}\right]}{\left(\frac{r}{\mu}\sqrt{\pi}\right)^{1-n}} = \frac{\lambda}{4\,\pi\varepsilon_0} \left( \frac{\mu^{2\,\varepsilon}}{\pi^{\varepsilon}\, r^{2\,\varepsilon}} \; \Gamma[\varepsilon] \right)$$

Problem:

- V[r] depends on artificial regulator μ
- We cannot remove the regulator μ:

$$V[r] \;\xrightarrow[\varepsilon\to 0]{}$$

*All physical quantities are independent of the regulator!!!*

# Dimensional Regularization Method

*All physical quantities R are independent of the regulator!!!*

$\Longrightarrow$ Renormalization Group Equation: $\dfrac{d\sigma}{d\mu} = 0$

Electric Field:

$$\mathbf{E} = \frac{-\partial V[r]}{\partial r} = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{2\,\varepsilon\,\mu^{2\varepsilon}\,\Gamma[\varepsilon]}{\pi^{\varepsilon}\,r^{1+2\varepsilon}}\right) \underset{\varepsilon\to 0}{\longrightarrow} \frac{\lambda}{2\pi\varepsilon_0\,r}$$

Energy ~ $\delta V$:

$$\delta V = V[r_1] - V[r_2] = \frac{\lambda}{4\pi\varepsilon_0}\,\mathrm{Log}\left[\frac{r_2^2}{r_1^2}\right]$$

*Same as before:*

Problem solved at the expense of an extra scale, $\mu$

*Different from before:*

Translational invariance symmetry preserved

Dimensional Regularization Respects Symmetries

# Renormalization

Expand V[r] in powers of ε:

$$V\ \ [r] = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{\varepsilon} + \ln\left[\frac{e^{-\gamma E}}{\pi}\right] + \ln\left[\frac{\mu^2}{r^2}\right] + O[\varepsilon]\right)$$

Let's invent a Minimal Subtraction (MS) prescription:

$$V_{MS}[r] = \frac{\lambda}{4\pi\varepsilon_0}\left(\qquad \ln\left[\frac{e^{-\gamma E}}{\pi}\right] + \ln\left[\frac{\mu^2}{r^2}\right] + O[\varepsilon]\right)$$

or even a Modified Minimal Subtraction (MS-bar) prescription:

$$V_{\overline{MS}}[r] = \frac{\lambda}{4\pi\varepsilon_0}\left(\qquad\qquad + \ln\left[\frac{\mu^2}{r^2}\right] + O[\varepsilon]\right)$$

After renormalization, we can remove regulator (ε→0), but we will still have μ-dependence and scheme dependence in V[r].

Again, physical observables are independent of μ and scheme,

$$(V_{\overline{MS}}[r_1] - V_{\overline{MS}}[r_2]) = (V_{MS}[r_1] - V_{MS}[r_2]) = \delta V$$

but <u>only</u> if you use a single scheme consistently.

$$(V_{\overline{MS}}[r_1] - V_{MS}[r_2]) = (V_{MS}[r_1] - V_{\overline{MS}}[r_2]) \neq \delta V$$

Mixed results introduce scheme dependence in physical observables.

# Renormalization Group Equation

Consider the basic parton factorization formula:

$$\sigma = \mathbf{f} \bullet \omega$$

The renormalization group equation reads:

$$\frac{d\sigma}{d\mu} = 0 = \frac{d\mathbf{f}}{d\mu}\,\omega + \mathbf{f}\,\frac{d\omega}{d\mu}$$

f and ω depend on μ, but σ does not!!!

After rearrangement (and a Mellin transform):

$$\frac{1}{\tilde{\mathbf{f}}}\,\frac{d\tilde{\mathbf{f}}}{d\,\mathbf{ln}[\mu]} = -\gamma = -\frac{1}{\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\,\mathbf{ln}[\mu]}$$

The anomalous dimension γ is the separation constant.

(If f obeyed exact scaling, γ would be zero--hence the term anomalous.)

The DGLAP evolution equation in moment (Mellin) space is:

$$\frac{d\tilde{\mathbf{f}}}{d\,\mathbf{ln}[\mu]} = -\gamma\,\tilde{\mathbf{f}}$$

or transformed back to x-space, is:

$$\frac{d\mathbf{f}}{d\,\mathbf{ln}[\mu]} = \mathbf{P} \bullet \mathbf{f}$$

The solution of the DGLAP evolution equation in moment (Mellin) space is:

$$\tilde{\mathbf{f}} \sim \mu^{-\gamma}$$

(Hence, the term anomalous dimension.)

# Recap

Regulator provides unique definition of V, f, $\omega$

- Cut off regulator, L:

  simple, but doesn't respect symmetries

- Dimensional regulator, $\varepsilon$:

  respects symmetries: translation, Lorentz, Gauge

  introduces new scale: $\mu$

All physical quantities (E, $\delta$V, $\sigma$) are independent of regulator

- Renormalization group equation: $d\sigma / d\mu = 0$

We can define renormalized quantities (V, f, $\omega$)

- Renormalized (V, f, $\omega$) are scheme dependent & arbitrary
- Physical quantities (E, $\delta$V, $\sigma$) are unique and

  scheme independent if we apply scheme consistently